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Infinite product exponents for modular forms

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Abstract

Recently, Choi obtained a description of the coefficients of the infinite product expansions of meromorphic modular forms over $\Gamma_0(N)$. Using this result, we provide some bounds on these infinite product coefficients for holomorphic modular forms. We give an exponential upper bound for the growth of these coefficients. We show that this bound is also a lower bound in the case that the genus of the associated modular curve $X_0(N)$ is 0 or 1.

1 Introduction

The study of modular forms is a rich and deep subject with connections to mathematics ranging from partitions to elliptic curves. In particular, examinations of modular forms are often informed by their divisors, most easily determined from the infinite product expansion of the modular form. Any holomorphic modular form $f(z) = \sum_{n=h}^{\infty} a(n)q^n$ where $q = e^{2\pi iz}$ and $a(h) = 1$ can be written as an infinite product

$$f(z) = q^h \prod_{m=1}^{\infty} (1 - q^m)^{c(m)}; \quad c(m) \in \mathbb{C}.$$

For example, consider the infinite product expansions of a family of such modular forms intimately connected to a modular form called the Dedekind η -function. The Dedekind η -function is a weight $1/2$ modular form defined by

$$\eta(z) = q^{1/24} \prod_{m=1}^{\infty} (1 - q^m).$$

One way to construct cusp forms of level N is through eta-quotients, expressions of the form

$$f(z) = \prod_{d|N} \eta(dz)^{r_d},$$

where r_d is some integer depending on the divisor d . Some holomorphic modular forms can be expressed as an eta-quotient, giving the values $c(m)$ in their infinite product expansion. However, only finitely many weight 2 newforms can be expressed as an eta-quotient, completely characterized by Ono and Martin (see [9]). Clearly, in this case, the values $c(m)$ are bounded.

However, $c(m)$ is not a repeating sequence (or even bounded) for generic modular forms $f(z)$. Understanding the function $c(m)$ can help determine the divisors of the associated

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modular forms. These appear in the theory of Borcherds products, which examines the rare case where the divisors of a form are supported at Heegner points. Our work makes use of divisors in general context. Bruinier et al. first derived an expression for $c(m)$ as a function of m for an associated meromorphic modular form defined over $SL_2(\mathbb{Z})$ (see [3]). Later work by Ahlgren [2] and Choi [4] generalized this result to meromorphic modular forms defined over $\Gamma_0(N)$. Related work by Movasati and Nikdelan also motivated this study. Further, Kohnen (see [7]) established a bound on the growth of $c(m)$ independent of these formulations if $f(z)$ has no zeros or poles on the upper half plane. He showed that given this condition, if $f(z)$ is a modular form for any finite index subgroup of $SL_2(\mathbb{Z})$, $c(m) \ll_f \log \log n \cdot \log n$ and that if $f(z)$ is a modular form for any congruence subgroup of $SL_2(\mathbb{Z})$, $c(m) \ll_f (\log \log n)^2$. Here we give tight bounds on $c(m)$ for some infinite classes of modular forms and an upper bound on the growth of $c(m)$ for any holomorphic modular form $f(z)$.

Throughout the paper, let $f(z) = \sum_{n=h}^{\infty} a(n)q^n$, $a(h) = 1$, be a holomorphic modular form of weight $k \in \mathbb{Z}_{\geq 0}$ over $\Gamma_0(N)$. Let \mathcal{F}_N be a fundamental domain for the action of $\Gamma_0(N)$ on the upper half plane $\mathcal{H} = \{x + iy \mid x, y \in \mathbb{R}, y > 0\}$. The infinite product expansion for $f(z)$ is written as

$$f(z) = q^h \prod_{m=1}^{\infty} (1 - q^m)^{c_f(m)}; \quad c_f(m) \in \mathbb{C}. \tag{1.1}$$

Theorem 1.1 *Assume the set of roots of $f(z)$ (with infinite product expansion as in (1.1)) in a fundamental domain \mathcal{F}_N is $\{z_j = x_j + iy_j\}_{j=1, \dots, r}$ with $y_1 \leq \dots \leq y_r$ and $r \geq 1$. Then we have that*

$$c_f(m) \ll \frac{e^{2\pi my_r}}{m^{3/2}}.$$

If the genus of $X_0(N)$ is 0 or 1, then we also obtain a lower bound on $c(m)$. In particular, we obtain an Ω bound, defined as follows. Given two arithmetic functions f, g defined on the natural numbers, $f = \Omega(g)$ implies that there exists some positive constant c such that for all $n_0 \in \mathbb{N}$ there exists infinitely many $n > n_0$ so that $f(n) \geq c \cdot g(n)$.

Theorem 1.2 *Assume the set of roots of $f(z)$ in \mathcal{F}_N is $\{z_j = x_j + iy_j\}_{j=1, \dots, r}$ with $y_1 \leq \dots \leq y_r$ and $r \geq 1$. If $f(z)$ is a modular form for $\Gamma_0(N)$ such that the genus of $X_0(N)$ is 0 or 1, then we have that*

$$c_f(m) = \Omega\left(\frac{e^{2\pi my_r}}{m^{3/2}}\right).$$

From the above results, we can also obtain a similar (slightly weaker) bound to that in [7] stated above.

Corollary 1.3 *Suppose that $f(z)$ is a modular form for $\Gamma_0(N)$ with no zeros or poles on the upper half plane. Then, we obtain $c_f(m) \ll \log m \cdot \log \log m$.*

2 Preliminaries

Let $q = e^{2\pi iz}$, with $z = x + iy \in \mathcal{H}$. Consider a modular form $f(z)$ with Fourier expansion $f(z) = \sum_{n=h}^{\infty} a(n)q^n$ with $a(h) = 1$ for a congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

Under the action of $\Gamma_0(N)$, two elements $z_1, z_2 \in \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ are equivalent, denoted $z_1 \sim z_2$, when there exists a $\sigma \in \Gamma_0(N)$ such that $\sigma z_1 = z_2$. Denote a set of representatives of the inequivalent cusps of $\Gamma_0(N)$ by \mathcal{C}_N , with $\mathcal{C}_N^* = \mathcal{C}_N \setminus \{\infty\}$. Then consider the modular curve of level N ,

$$X_0(N) = \Gamma_0(N) \backslash (\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})).$$

Let $v_z^{(N)}(f(z))$ be the (weighted) order of the zero of $f(z)$ at z on $X_0(N)$. Now, we will define a meromorphic modular form of weight 2 for $\Gamma_0(N)$ that frequently appears in these studies (see [2, 3]):

Proposition 2.1 *Let $f(z)$ be a modular form for $\Gamma_0(N)$. Define*

$$f_\theta(z) := \frac{\theta f(z)}{f(z)} + \frac{k/12 - h}{N - 1} \cdot NE_2(Nz) + \frac{h - Nk/12}{N - 1} \cdot E_2(z),$$

where

$$\theta f(z) = \sum_{n=h}^{\infty} na_n q^n$$

is the Ramanujan θ operator,

$$E_2(z) = 1 - 24 \sum_{n \geq 1} \sigma_1(n)q^n$$

is the normalized quasimodular Eisenstein series of weight $k = 2$, and $\sigma_j(n) = \sum_{d|n} d^j$ is the sum of the j th powers of divisors of n . Then $f_\theta(z)$ is a meromorphic modular form of weight 2 for $\Gamma_0(N)$.

Consider the weight 0 index m Poincaré series (Theorem 1 in [10]) for all $z \in \mathcal{H}$ and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$:

$$F_{N,m}(z, s) = \sum_{\gamma \in \Gamma_0(N)_\infty \backslash \Gamma_0(N)} \pi \sqrt{|\text{Im}(\gamma z)|} I_{s-\frac{1}{2}}(|2\pi m \text{Im}(\gamma z)|) e^{-2\pi i m \text{Re}(\gamma z)}, \tag{2.1}$$

where $I_\nu(x)$ is the usual modified I -Bessel function of order ν .

Proposition 2.2 (§1 in [4]) *Define $j_{N,m}(z)$ to be the analytic continuation of $F_{N,m}(z, s)$ (where $\text{Re}(s) \leq 1$) as $s \rightarrow 1^+$. Then, $j_{N,m}(z)$ is the constant term of the Fourier expansion of $F_{N,m}(z, 1)$ when $t \in \mathcal{C}_N$.*

Next we define the differential operator ξ_0 that plays an important role in the study of Harmonic Maass forms [12], defined in the next section.

Proposition 2.3 ([12, p. 30]) *Define the differential operator ξ_0 on the space of functions $j_{N,m}$*

$$\xi_0(j_{N,m}) := 2i \overline{\frac{\partial}{\partial \bar{z}}} j_{N,m}(z).$$

The differential operator ξ_0 maps $j_{N,d}$ to a cusp form.

Here, we consider the regularized integral (since $f_\theta(z)$ is generally not holomorphic on \mathcal{H})

$$\int_{\mathcal{F}_N}^{reg} f_\theta(z) \cdot \xi_0(j_{N,m}(z)) dx dy$$

where the regularization method can be found in [4].

3 Proof of main result

Given this setup, the exponents of the infinite product expansion of a modular form can be obtained by applying the Möbius inversion formula to the result obtained in [4].

Proposition 3.1 Consider a normalized (holomorphic) modular form $f(z)$ of weight k on $\Gamma_0(N)$ for $N > 1$ with infinite product expansion as in (1.1). Then, we have that

$$c_f(m) = \frac{1}{m} \sum_{d|m} \mu\left(\frac{m}{d}\right) \left(\sum_{z \in \mathcal{F}_N \cup C_N^*} v_z^{(N)}(f) j_{N,d}(z) - \int_{\mathcal{F}_N}^{reg} f_\theta(z) \cdot \xi_0(j_{N,d}(z)) dx dy \right) + \begin{cases} \frac{2Nk-24h}{N-1} & N \nmid m \\ 2k & N | m \end{cases}$$

where μ is the Möbius function.

A normalized modular form with integral coefficients $f(z) = q^h + \sum_{n=h+1}^\infty a(n)q^n$ has a product expansion $f(z) = q^h \prod_{m=1}^\infty (1 - q^m)^{c_f(m)}$ where the $c_f(m)$ are integers. This can be seen by expanding the infinite product and solving for the coefficients of the Fourier expansion. Now we consider $c_f(m)$ as $m \rightarrow \infty$ by computing the growth of terms in the above expression.

3.1 Growth of $j_{N,m}(z)$

We note that $F_{N,m}(z, s)$ is a harmonic Maaßform (defined below) and cite a Lemma giving an explicit computation for the values for the analytic continuation $j_{N,m}(z)$.

Definition 3.2 (Definition 7.1 [12]) A smooth function $f : \mathcal{H} \rightarrow \mathbb{C}$ is a weak harmonic Maaß form of weight k for $\Gamma_0(N)$ if the following conditions are satisfied:

- (1) f transforms like a modular form under the action of $\Gamma_0(N)$,
- (2) f is in the kernel of the weight k hyperbolic Laplacian

$$\Delta_k = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

- (3) f has at most linear exponential growth at the cusps of f .

Lemma 3.3 (Theorem 2.1[4]) The function $F_{N,m}(z, 1)$ is a weak harmonic Maaß form of weight 0 on $\Gamma_0(N)$. Further at $F_{N,m}(z, 1)$ has the following properties at each cusp t : If $t \sim \infty$,

$$F_{N,m}(z, 1) = q^{-m} + \sum_{n \geq 0} b_m(n, 1)q^n + \sum_{n > 0} b_m(-n, 1)e^{2\pi in\bar{z}}$$

and otherwise if $t \approx \infty$,

$$\lim_{\text{Im } z \rightarrow \infty} F_{N,m}(\sigma_t z, 1) = j_{N,m}(t)$$

where $\sigma_t \in \text{SL}_2(\mathbb{R})$ is defined so that $\sigma_t^{-1}\infty = t$ and $\sigma_t \Gamma_0(N)_t \sigma_t^{-1} = \left\{ \pm \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \mid n \in \mathbb{Z} \right\}$.

Lemma 3.4 Let $z = x + iy \in \mathcal{F}_N \cup C_N^*$. As $m \rightarrow \infty$, we have that $j_{N,m}(z) \asymp \left(\frac{e^{2\pi my}}{\sqrt{m}} \right)$.

Proof From [10], we can obtain the analytic continuation of $F_{N,m}(z, s)$ where $\text{Re}(s) > 1/2$ and a Fourier expansion for this function as $s \rightarrow 1^+$ which we term $F_{N,m}(z, 1)$. Since $z \approx \infty$, $j_{N,m}(z)$ is the constant term of this Fourier expansion of $F_{N,m}(z, 1)$ and can be computed as

$$j_{N,m}(z) = e^{2\pi i|m|x} \sqrt{y} I_{1/2}(2\pi|m|y) + a_m(1) \sqrt{y} K_{1/2}(2\pi|m|y),$$

where $a_m(1)$ is the m th coefficient of the Maaß-Eisenstein series for $\Gamma_0(N)$ and $K_\nu(y), I_\nu(y)$ are the modified Bessel functions of the first and second kind with special values given above. Then, apply the relations on these modified Bessel functions, noting that for $y \in \mathbb{R} \setminus \{0\}$, (see 10.2.13 and 10.2.17 in [1]),

$$I_{1/2}(y) = \frac{\sinh |y|}{\sqrt{\frac{\pi|y|}{2}}}; \quad K_{1/2}(y) = \frac{e^{-|y|}}{\sqrt{\frac{\pi|y|}{2}}}.$$

Using these expressions we evaluate:

$$j_m(z) = \frac{1}{\pi \sqrt{m}} \left(e^{2\pi imx} \sinh(2\pi my) + a_n(1) e^{-2\pi my} \right).$$

If $z \in \mathcal{C}_N^*$, $y = 0$, and thus $j_m(z) \asymp \frac{1}{\sqrt{m}} \ll 1$. Else if $z \in \mathcal{H}$, $y > 0$. Taking $m \rightarrow \infty$, and simplifying the above expression, we retrieve the desired result:

$$\lim_{m \rightarrow \infty} j_m(z) \asymp \frac{e^{2\pi my}}{\sqrt{m}}.$$

□

3.2 Regularized integration

We examine the growth of the regularized integral as in Proposition 3.1:

$$R(m) = \int_{\mathcal{F}_N}^{reg} f_\theta(z) \cdot \xi_0(j_{N,m}(z)) dx dy. \tag{3.1}$$

First, we compute some cases where this term vanishes, so that the contribution from this term in the growth of $c_f(m)$ is 0. The following lemma describes this.

- Lemma 3.5** (1) *If the genus of $X_0(N)$ is 0, the regularized integral as defined above vanishes.*
 (2) *If genus of $\Gamma_0(N)$ is 1, the regularized integral vanishes for infinitely many positive integers m .*

Proof Recall that $\xi_0(j_{N,d}(z))$ is a weight two cusp form. Suppose that the genus of $X_0(N)$ is zero. Then the space of cusp forms of weight two is trivial, so $\xi_0(j_{N,d}(z))$ must be 0. It follows that the regularized integral vanishes.

Now let the genus of $X_0(N)$ be one. The following argument appears as a remark in [4]. The space of weight two cusp forms of weight two is spanned by a unique normalized weight 2 cusp form $g(z) = \sum_{n=1} a(n)q^n$. There exists an elliptic curve E_g of conductor dividing N whose Hasse-Weil L -series coincides with L -function for $g(z)$. In other words, for all values of p that do not divide N ,

$$p + 1 - a(p) = \#E_g/\mathbb{F}_p.$$

If E_g is supersingular at p , the number of points of $E_g/\mathbb{F}(p)$ is exactly $p + 1$ and $a(p) = 0$. So we have that for an odd m , $j_{N,m}(z)$ identically vanishes if and only if E_g is supersingular at some $p|m$. Since there exist infinitely many supersingular primes for every elliptic curve over \mathbb{Q} (see [5]), there are infinitely many positive integers m such that $j_{N,m}(z)$ is holomorphic on \mathcal{H} (i.e. $\xi_0 j \equiv 0$). \square

This regularized integral $R(m)$ may not vanish if the genus of $X_0(N)$ is not 0. Below we address the growth of $R(m)$. First, we characterize the growth of the Fourier coefficient $b_m(1, 0)$ of the weight 0 Poincaré series of index m .

Lemma 3.6 Recall the weight 0 and index m Poincaré series given in Eq. 2.1 with Fourier expansion

$$F_{N,m}(z, s) = q^{-m} + \sum_{n \geq 0} b_m(n, 1)q^n + \sum_{n > 0} b_m(-n, 1)e^{2\pi in\bar{z}}.$$

Then $b_m(1, 0) \ll m^{1/4} e^{4\pi\sqrt{m}}$.

Proof Theorem 8.4 in [12] gives the following expansion of $b_m(1, 0)$:

$$b_m(1, 0) = 2\pi (m)^{1/2} \sum_{c > 0, N|c} \frac{K_0(-m, 1, c)}{c} I_1\left(\frac{4\pi\sqrt{|m|}}{c}\right),$$

where $K_0(-m, n, c)$ is the Kloosterman sum defined as

$$K(a, b; m) = \sum_{0 \leq d \leq m-1, \gcd(d, m)=1} e^{\frac{2\pi i}{m}(ad+bd^*)},$$

where d^* is the multiplicative inverse of d modulo m . Weil’s bound gives

$$b_m(1, 0) \ll m^{1/2} \sum_{c > 0, N|c} \frac{c^{1/2+\epsilon}}{c} I_1\left(\frac{4\pi\sqrt{|m|}}{c}\right).$$

Now we expand the modified Bessel function $I_1\left(\frac{2\pi\sqrt{|m|}}{c}\right)$.

$$I_1\left(\frac{4\pi\sqrt{|m|}}{c}\right) = \frac{4\pi\sqrt{|m|}}{c} \sum_{k=0}^{\infty} \frac{(\pi^2|m|/c^2)^k}{k!(1+k)!}.$$

We can now examine the growth I_1 of $b_m(1, 0)$ as $m \rightarrow \infty$. The Bessel function is approximated by

$$I_1\left(\frac{4\pi\sqrt{|m|}}{c}\right) \sim \frac{e^{(4\pi\sqrt{|m|})/c}}{\sqrt{8\pi^2\sqrt{|m|}/c}}$$

for sufficiently large m (see 10.41 in [11]). Then as $m \rightarrow \infty$, we obtain

$$b_m(1, 0) \ll m^{1/4} e^{4\pi\sqrt{m}}.$$

\square

Lemma 3.7 Consider the regularized integral in 3.1. Suppose that the set of zeros of $f(z)$ in a chosen fundamental domain is $\{z_j = x_j + iy_j\}_{j=1,\dots,r}$ with $y_1 \leq \dots \leq y_r$, $r \geq 1$. As $m \rightarrow \infty$ we can bound $R(m)$ as:

$$R(m) \ll \frac{e^{2\pi my_r}}{\sqrt{m}}.$$

Proof The function $f_\theta(z)$ is a meromorphic modular form of weight two on $\Gamma_0(N)$, which is holomorphic at each cusp and each pole is simple. Thus, it satisfies the conditions for Lemma 3.1 in [4], giving the following expansion for $R(m)$.

$$R(m) = \lim_{\epsilon \rightarrow 0} \int_{\mathcal{F}_N(f_\theta, \epsilon)} f_\theta(z) \cdot \xi_0(j_{N,m}(z)) dx dy \tag{3.2}$$

$$= b_m(1, 0)a(0) + a(m) + \sum_{t \in C_N^*} \alpha_t f_\theta(t) j_{N,m}(t) + \sum_{t \in S(f_\theta)} \frac{2\pi i}{\ell_t} \text{Res}_t(f_\theta) j_{N,m}(t) \tag{3.3}$$

where α_t and ℓ_t are constants depending on t defined in [4]. Lemma 3.6 shows that term $b_m(1, 0)a(0) \ll m^{1/4} e^{4\pi\sqrt{m}}$. Moreover, the term $a(m)$ is negligible, since the coefficients of the modular form $f(z)$ has polynomial growth in m . Now, the cusps in C_N^* are rational points. From the proof of Lemma 3.4, we have

$$\lim_{m \rightarrow \infty} j_m(z) \asymp \frac{e^{2\pi my_r}}{\sqrt{m}}.$$

For each $t \in C_N^*$, $\text{Im}(t) = 0$, so this gives

$$\sum_{t \in C_N^*} \alpha_t f_\theta(t) j_{N,m}(t) \asymp \frac{1}{\sqrt{m}}. \tag{3.4}$$

Now we examine the last term in this expression, a sum over the poles of $f_\theta(z)$. These are the zeros of $f(z)$. Since the m -dependence of this term also comes from $j_{N,m}(t)$, the growth of this term is dominated by the zeros of $f(z)$. Thus we have

$$\sum_{t \in S(f_\theta)} \frac{2\pi i}{\ell_t} \text{Res}_t(f_\theta) j_{N,m}(t) \asymp \frac{e^{2\pi imy_r}}{\sqrt{m}}. \tag{3.5}$$

as in the proof of Lemma 3.4. This dominates the term in (3.4), so the growth of $R(m)$ is dictated by the above. Thus, given the integral expansion in (3.3),

$$R(m) \ll \frac{e^{2\pi my_r}}{\sqrt{m}}.$$

□

3.3 Proof of results

We begin by proving the second theorem since many of the arguments used to prove the first theorem parallel the ones employed here:

Proof of Theorem 1.2 From Proposition 3.1, we obtain

$$c_f(m) = \frac{1}{m} \sum_{d|m} \mu\left(\frac{m}{d}\right) \left(\sum_{z \in \mathcal{F}_N \cup C_N^*} v_z^{(N)}(f(z)) j_{N,d}(z) - \int_{\mathcal{F}_N}^{reg} f_\theta(z) \cdot \xi_0(j_{N,d}(z)) dx dy \right) + \begin{cases} \frac{2Nk-24h}{N-1} & N \nmid m \\ 2k & N | m. \end{cases}$$

According to Lemma 3.5, since the genus of $X_0(N)$ is 0 or 1, the regularized integral defined in Sect. 3.2 vanishes infinitely often. Thus, for infinitely many m , we have

$$c_f(m) = \Omega \left(\frac{1}{m} \sum_{d|m} \mu\left(\frac{m}{d}\right) \left(\sum_{z \in \mathcal{F}_N \cup C_N^*} v_z^{(N)}(f(z)) j_{N,d}(z) \right) + \begin{cases} \frac{2Nk-24h}{N-1} & N \nmid m \\ 2k & N | m \end{cases} \right)$$

Note that the term depending on whether $N|m$ or not is bounded with respect to m . Then, since m dominates the other divisors of m as $m \rightarrow \infty$, we evaluate the expression $\sum_{z \in \mathcal{F}_N \cup \mathcal{C}_N^*} v_z^{(N)}(f(z))j_{N,d}(z)$ only at $d = m$. Further $v_z^{(N)}$ is a constant that does not depend on m . Thus we arrive at

$$c_f(m) = \Omega \left(\frac{1}{m} \sum_{z \in \mathcal{F}_N \cup \mathcal{C}_N^*} j_{N,m}(z) \right).$$

By Lemma 3.4, we have $j_{N,m}(z) \asymp \frac{e^{2\pi my}}{\sqrt{m}}$ for $z = x + iy$ in $\mathcal{F}_N \cup \mathcal{C}_N^*$. Then, the dominating term of the sum is the term with $z = z_r$, giving

$$c_f(m) = \Omega \left(\frac{e^{2\pi my_r}}{m^{3/2}} \right).$$

□

Now, suppose we relax the conditions on the genus of $X_0(N)$. We now prove the upper bound on $c_f(m)$ in general given in Theorem 1.1.

Proof of Theorem 1.1 We examine the worst case growth of the equation

$$c_f(m) = \frac{1}{m} \sum_{d|m} \mu \left(\frac{m}{d} \right) \left(\sum_{z \in \mathcal{F}_N \cup \mathcal{C}_N^*} v_z^{(N)}(f(z))j_{N,d}(z) - \int_{\mathcal{F}_N}^{reg} f_\theta(z) \cdot \xi_0(j_{N,d}(z)) dx dy \right) + \begin{cases} \frac{2Nk-24h}{N-1} & N \nmid m, \\ 2k & N|m. \end{cases}$$

Like above, the term depending on whether $N|m$ or not is bounded as $m \rightarrow \infty$. Evaluating the remaining expression at $d = m$ we have

$$c_f(m) \ll \frac{1}{m} \left(\sum_{z \in \mathcal{F}_N \cup \mathcal{C}_N^*} v_z^{(N)}(f(z))j_{N,m}(z) - \int_{\mathcal{F}_N}^{reg} f_\theta(z) \cdot \xi_0(j_{N,m}(z)) dx dy \right) + O(1).$$

In the worst case, the regularized integral is $O\left(\frac{e^{2\pi my_r}}{\sqrt{m}}\right)$, by Lemma 3.7. Combining this with Lemma 3.4, we obtain

$$c_f(m) \ll \frac{1}{m} \left(\frac{e^{2\pi my_r}}{\sqrt{m}} \right).$$

□

Finally, we see that the bounds yielded by our calculation are consistent with earlier work in [7].

Proof of Corollary 1.3 Note that as $m \rightarrow \infty$, we have the bound

$$c_f(m) \asymp \frac{1}{m} \sum_{d|m} \left(\sum_{z \in \mathcal{F}_N \cup \mathcal{C}_N^*} v_z^{(N)}(f(z))j_{N,d}(z) - \int_{\mathcal{F}_N}^{reg} f_\theta(z) \cdot \xi_0(j_{N,d}(z)) dx dy \right) + O(1).$$

Since $f(z)$ has no roots or poles on the upper half plane, the first term in the above summation,

$$\sum_{z \in \mathcal{F}_N \cup \mathcal{C}_N^*} v_z^{(N)}(f(z))j_{N,d}(z)$$

reduces to

$$\sum_{z \in C_N^*} v_z^{(N)}(f(z))j_{N,d}(z).$$

Here we exclude the cusp at ∞ . By Lemma 3.4, the above expression grows asymptotically proportional to $e^{2\pi my_r} / \sqrt{m}$, where y_r is the largest imaginary part of a summand with nonzero order. However, all of the cusps $z \approx \infty$ are on the rational line and thus have imaginary part 0. Consequently,

$$\sum_{z \in C_N^*} v_z^{(N)}(f(z))j_{N,d}(z) \asymp_{m \rightarrow \infty} \frac{1}{\sqrt{m}} \rightarrow 0.$$

Now consider the regularized integral. As per Theorem 1.4 in [4],

$$[f_\theta(z)]_d + \int_{\mathcal{F}_N}^{reg} f_\theta(z) \cdot \xi_0(j_{N,d}(z)) dx dy = \sum_{z \in C_N^*} v_z^{(N)}(f(z))j_{N,d}(z),$$

where $[f_\theta(z)]_d$ is the coefficient of q^d in the Fourier expansion of $f_\theta(z)$. Using the above bound on the cusp summation,

$$\int_{\mathcal{F}_N}^{reg} f_\theta(z) \cdot \xi_0(j_{N,d}(z)) dx dy \ll [f_\theta(z)]_d.$$

Now recall the definition

$$f_\theta(z) = \frac{\theta f}{f} + \frac{k/12 - h}{N - 1} \cdot NE_2(Nz) + \frac{h - Nk/12}{N - 1} E_2(z).$$

Since $\frac{\theta f}{f}$ is holomorphic and quasi-modular of weight 2, the coefficients of the Fourier expansion of $\frac{\theta f}{f}$ can be bounded by $[\frac{\theta f}{f}]_d \ll d \log d$, (see [8]). Given the definition of the Eisenstein series, we can also bound the coefficients of $E_2(z)$ (and $\sigma_1(d)$ as in [6]):

$$[E_2(z)]_d \asymp \sigma_1(d) \ll n \log \log n.$$

Combining these two bounds gives

$$\int_{\mathcal{F}_N}^{reg} f_\theta(z) \cdot \xi_0(j_{N,d}(z)) dx dy \ll [f_\theta(z)]_d \ll d \log d + d \log \log d \ll d \log d.$$

Then, we can once again apply $\sigma_1(n) \ll n \log \log n$ to the summation over $d|m$ to compute the desired bound for $c_f(m)$:

$$c_f(m) \ll \frac{1}{m} ((m \log \log m) \cdot \log m) + O(1) \ll \log m \cdot \log \log m.$$

□

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