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A small improvement in the gaps between consecutive zeros of the Riemann zeta-function

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Abstract

Feng and Wu introduced a new general coefficient sequence into Montgomery and Odlyzko's method for exhibiting irregularity in the gaps between consecutive zeros of $\zeta(s)$ assuming the Riemann hypothesis. They used a special case of their sequence to improve upon earlier results on the gaps. In this paper we consider a general sequence related to that of Feng and Wu, and introduce a somewhat less general sequence $\{a_n\}$ for which we write the Montgomery–Odlyzko expressions explicitly. As an application, we give the following slight improvement of Feng and Wu's result: infinitely often consecutive non-trivial zeros of the Riemann zeta-function differ by at most 0.515396 times the average spacing and infinitely often they differ by at least 2.7328 times the average spacing.

Keywords: Riemann zeta function, Zeros, Critical line, Gaps

Mathematics Subject Classification: Primary 11M26; Secondary 11M06

Contents

1 Background

It is well known that the Riemann zeta-function $\zeta(s)$ has infinitely many nontrivial zeros $s = \rho = \beta + i\gamma$, and all of them are in the critical strip $0 < \mathrm{Re}\, s = \sigma < 1$, $-\infty < \mathrm{Im}\, s = t < \infty$.

If $N(T)$ denotes the number of zeros $\rho = \beta + i\gamma$ (β and γ real), for which $0 < \gamma \le T$, then

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right),$$

with

$$S(T) = \frac{1}{\pi}\arg\zeta\left(\frac{1}{2} + iT\right)$$

and

$$S(T) = O(\log T).$$

This is the Riemann–von Mangoldt formula for $N(T)$. Hence, if we let $0 < \gamma \le \gamma'$ denote consecutive ordinates of non-trivial zeros of $\zeta(s)$, the average size of $\gamma' - \gamma$ is $\gamma/N(\gamma) \sim 2\pi/\log\gamma$. Let

$$\lambda = \limsup_{\gamma>0}(\gamma' - \gamma)\frac{\log\gamma}{2\pi}$$

and

$$\mu = \liminf_{\gamma>0}(\gamma' - \gamma)\frac{\log\gamma}{2\pi}.$$

We note that $\mu \le 1 \le \lambda$ and it is expected that $\mu = 0$ and $\lambda = +\infty$. The problem of studying λ and μ is important for a number of reasons. Montgomery and Weinberger [11] discovered an effect of the exceptional zero of an L-function on the spacing of zeros, which led Montgomery [9] to the Pair Correlation Conjecture (see also Conrey and Iwaniec [4]): for any fixed $0 < \alpha < \beta$,

$$\sum_{\substack{0<\gamma,\gamma'\le T\\ \frac{2\pi\alpha}{\log T}\le\gamma'-\gamma\le\frac{2\pi\beta}{\log T}}} 1 \sim N(T)\int_\alpha^\beta \left(1-\left(\frac{\sin\pi u}{\pi u}\right)^2\right)du.$$

The paper [9] also contains explicit bound $\mu \le 0.68$ conditionally on the Riemann hypothesis.

Let $N_0(T)$ be the number of zeros of $\zeta\left(\frac{1}{2}+it\right)$ when $0 < t \le T$, each zero counted with multiplicity. The Riemann hypothesis is the conjecture that $N_0(T) = N(T)$.

In this note we prove the following theorem.

Theorem 1 *Assume the Riemann hypothesis. Then we have*

$$\mu < 0.515396$$

and

$$\lambda > 2.7328.$$

We briefly describe the history of the problem, focusing mainly on μ.

- [12]: in 1946 Selberg remarked that $\mu < 1 < \lambda$ unconditionally.

Now suppose that T is a large real number and $K = T(\log T)^{-2}$. Let

$$h(c) = c - \frac{\operatorname{Re}\left(\sum_{nk\le K} a_k\overline{a_{nk}}g_c(n)\Lambda(n)n^{-1/2}\right)}{\sum_{k\le K}|a_k|^2}, \tag{1}$$

where

$$g_c(n) = \frac{2\sin\left(\pi c\frac{\log n}{\log T}\right)}{\pi\log n}$$

and Λ is the von Mangoldt's function.

In the following results, the truth of the Riemann hypothesis is assumed.

- [10]: in 1981 by an argument using the Guinand–Weil explicit formula, Montgomery and Odlyzko showed that if $h(c) < 1$ for some choice of c and $\{a_n\}$, then $\lambda \ge c$, and if $h(c) > 1$ for some choice of c and $\{a_n\}$, then $\mu \le c$. They used the coefficients

$$a_k = \frac{1}{k^{\frac{1}{2}}}f\left(\frac{\log k}{\log K}\right) \quad \text{and} \quad a_k = \frac{\lambda(k)}{k^{\frac{1}{2}}}f\left(\frac{\log k}{\log K}\right),$$

where f is a continuous function of bounded variation, and $\lambda(k)$ is the Liouville function. With this choice of the coefficients they obtained $\lambda > 1.9799$ and $\mu < 0.5179$ by optimizing over the functions f.

- [3]: in 1984 Conrey et al. chose the coefficients

$$a_k = \frac{d_r(k)}{\sqrt{k}} \quad \text{and} \quad a_k = \frac{\lambda(k) d_r(k)}{\sqrt{k}},$$

where $d_r(k)$ is a multiplicative function defined on integral powers of a prime p by

$$d_r(p^k) = \frac{\Gamma(k+r)}{\Gamma(r)k!}.$$

The choice $r = 1.1$ with the latter a_k yields $\mu < 0.5172$ and the choice $r = 2.2$ with the former a_k yields $\lambda > 2.337$.

- [7]: in 2005, by making use of Wirtinger's inequality and the asymptotic formulae for the fourth mixed moments of the zeta-function and its derivative, Hall proved that $\lambda > 2.6306$.

- [2]: in 2010, Bui et al. considered the coefficients of the form

$$a_k = \frac{d_r(k)}{\sqrt{k}} f\left(\tfrac{\log K/k}{\log K}\right) \quad \text{and} \quad a_k = \frac{\lambda(k) d_r(k)}{\sqrt{k}} f\left(\tfrac{\log K/k}{\log K}\right)$$

for a polynomial f and obtained $\lambda > 2.69$ and $\mu < 0.5155$.

- [5]: in 2012, Feng and Wu introduced the coefficient

$$\begin{aligned} a_k = \frac{d_r(k)}{k^{\frac{1}{2}}} \Bigg(& f_1\left(\frac{\log K/k}{\log K}\right) + f_2\left(\frac{\log K/k}{\log K}\right) \sum_{p_1 p_2 | k} \frac{\log p_1 \log p_2}{\log^2 K} \\ & + f_3\left(\frac{\log K/k}{\log K}\right) \sum_{p_1 p_2 p_3 | k} \frac{\log p_1 \log p_2 \log p_3}{\log^3 K} + \cdots \\ & + f_I\left(\frac{\log K/k}{\log K}\right) \sum_{p_1 p_2 \cdots p_I | k} \frac{\log p_1 \log p_2 \cdots \log p_I}{\log^I K} \Bigg), \end{aligned}$$

for any integer $I \geq 2$. Using $I = 2$ they obtained $\lambda > 2.7327$ and $\mu < 0.5154$, or, to higher precision, $\lambda > 2.73272$ and $\mu < 0.515398$.

We remark that for $I = 2$ the Feng and Wu coefficient is equivalent to

$$\begin{aligned} a_k = \frac{d_r(k)}{k^{\frac{1}{2}}} \Bigg(& f_1\left(\frac{\log K/k}{\log K}\right) + f_2\left(\frac{\log K/k}{\log K}\right) \sum_{p_1 | k} \frac{\log^2 p_1}{\log^2 K} \\ & + f_3\left(\frac{\log K/k}{\log K}\right) \sum_{p_1 | k} \frac{\log^3 p_1}{\log^3 K} + \cdots + f_I\left(\frac{\log K/k}{\log K}\right) \sum_{p_1 | k} \frac{\log^I p_1}{\log^I K} \Bigg), \end{aligned}$$

for which the calculations are simpler.

To prove Theorem 1, we choose the coefficients

$$a_k = \frac{\lambda(k) d_r(k)}{k^{\frac{1}{2}}} f_1\left(\frac{\log K/k}{\log K}\right) + \frac{\lambda(k) d_r(k)}{k^{\frac{1}{2}}} \sum_{p|k} P\left(\frac{\log p}{\log K}\right) \tilde{f}_1\left(\frac{\log K/k}{\log K}\right),$$

where $f_1, \tilde{f}_1, P$ are some polynomials to be chosen later. These a_k are less general than

$$a_k = \frac{\lambda(k) d_r(k)}{k^{\frac{1}{2}}} \left(f_1\left(\frac{\log K/k}{\log K}\right) + f_2\left(\frac{\log K/k}{\log K}\right) \sum_{p_1|k} \frac{\log^2 p_1}{\log^2 K} \right.$$
$$+ f_3\left(\frac{\log K/k}{\log K}\right) \sum_{p_1|k} \frac{\log^3 p_1}{\log^3 K} + \cdots$$
$$\left. + f_I\left(\frac{\log K/k}{\log K}\right) \sum_{p_1|k} \frac{\log^I p_1}{\log^I K} \right),$$

but the former sequence is simpler, so we are able to write the Montgomery–Odlyzko expressions for it explicitly.

As for the limitations of the employed method, in [3] it is shown that $h(c) < 1$ if $c < \frac{1}{2}$ and $h(c) > 1$ if $c \geq 6.2$ (the authors note that the latter bound can be improved to $h(c) > 1$ if $c \geq 3.74$) and the length K is $\leq T$. So the value of $\mu = \frac{1}{2}$ is not attainable with any sequence $\{a_k\}$ using this method. One may try to use the form of the coefficients $\{a_k\}$ of the present paper in the method of [6] obtaining $\lambda > 3.072$ on the Generalized Riemann hypothesis, but this method is more technical and involved, and the numerical calculations seem to require more computational resources. The best known bound for λ just assuming the Riemann hypothesis is $\lambda > 2.9$ due to Bui [1]. It is possible that Feng and Wu's result $\lambda > 3.072$ can also be obtained just assuming the Riemann hypothesis. For another application of Feng's mollifier, see [8] and the references therein.

2 Lemmas

Lemma 1 *Let a_i be integer for $1 \leq i \leq m$, $D > 1$ and f is a continuous function. Then*

$$\int_1^D \frac{\log^{a_1-1} x_1}{x_1} dx_1 \int_1^{\frac{D}{x_1}} \frac{\log^{a_2-1} x_2}{x_2} dx_2 \cdots \int_1^{\frac{D}{x_1 x_2 \cdots x_m}} \frac{f(x_1 x_2 \cdots x_m x)}{x} dx$$
$$= \frac{\prod_{i=1}^m (a_i - 1)!}{(\sum_{i=1}^m a_i)!} \int_1^D \frac{f(x) \log^{\sum_{i=1}^m a_i} x}{x} dx.$$

Lemma 2 *Let a_i be integer for $1 \leq i \leq m$, and g is a polynomial. Then we have*

$$\sum_{k \leq K} \frac{d_r(k)^2}{k} g\left(\frac{\log K/k}{\log K}\right) = A_r r^2 \int_1^K g\left(\frac{\log K/x}{\log K}\right) (\log x)^{r^2-1} \frac{dx}{x}$$
$$+ O\left((\log K)^{r^2-1}\right)$$

and

$$\sum_{p_1 p_2 \cdots p_m \leq K} \prod_{i=1}^m \frac{\log^{a_i} p_i}{p_i} \mu^2(p_1 p_2 \cdots p_m) \sum_{k_0 \leq K/(p_1 p_2 \ldots p_m)} \frac{d_r(k_0)^2}{k_0} g\left(\frac{\log K/(p_1 p_2 \ldots p_m k_0)}{\log K}\right)$$
$$= (1 + O(\log^{-1} K)) A_r r^2 \int_1^K \log^{a_1-1} x_1 \frac{dx_1}{x_1} \int_1^{\frac{K}{x_1}} \log^{a_2-1} x_2 \frac{dx_2}{x_2} \cdots$$
$$\times \int_1^{\frac{K}{x_1 x_2 \cdots x_{m-1}}} \log^{a_m-1} x_m \frac{dx_m}{x_m} \int_1^{\frac{K}{x_1 x_2 \cdots x_m}} g\left(\frac{\log K/(x_1 x_2 \cdots x_m x)}{\log K}\right) (\log x)^{r^2-1} \frac{dx}{x},$$

where A_r is a constant that depends only on r.

For the proof of Lemmas 1 and 2, see [5].

3 Proof of Theorem 1

To give an upper bound for μ, we evaluate $h(c)$ in (1) with the coefficients

$$a_k = \frac{\lambda(k)d_r(k)}{k^{\frac{1}{2}}} f_1\left(\frac{\log K/k}{\log K}\right) + \frac{\lambda(k)d_r(k)}{k^{\frac{1}{2}}} \sum_{p|k} P\left(\frac{\log p}{\log K}\right) \tilde{f}_1\left(\frac{\log K/k}{\log K}\right),$$

where $r \geq 1$ and $f_1, \tilde{f}_1, P$ are polynomials.

First, we evaluate the denominator in the ratio in the definition of $h(c)$.

$$\begin{aligned}\sum_{k\leq K} |a_k|^2 = &\sum_{k\leq K} \frac{d_r(k)^2}{k} f_1\left(\frac{\log K/k}{\log K}\right)^2 \\ &+2\sum_{k\leq K} \frac{d_r(k)^2}{k} f_1\left(\frac{\log K/k}{\log K}\right)\tilde{f}_1\left(\frac{\log K/k}{\log K}\right)\sum_{p|k} P\left(\frac{\log p}{\log K}\right) \\ &+\sum_{k\leq K} \frac{d_r(k)^2}{k} \tilde{f}_1\left(\frac{\log K/k}{\log K}\right)^2 \sum_{p|k} P\left(\frac{\log p}{\log K}\right)\sum_{q|k} P\left(\frac{\log q}{\log K}\right) \\ = &\tilde{D}_1 + \tilde{D}_2 + \tilde{D}_3.\end{aligned}$$

Using Lemma 2 and recalling that $K = T(\log T)^{-2}$, we have

$$\begin{aligned}\tilde{D}_1 &= A_r r^2 \int_1^K f_1\left(\frac{\log K/x}{\log K}\right)^2 (\log x)^{r^2-1}\frac{dx}{x} + O\left((\log T)^{r^2-1}\right) \\ &= A_r r^2 (\log K)^{r^2} \int_0^1 (1-u)^{r^2-1} f_1(u)^2 du + O\left((\log T)^{r^2-1}\right) \\ &= A_r r^2 (\log T)^{r^2} \int_0^1 (1-u)^{r^2-1} f_1(u)^2 du + O\left((\log T)^{r^2-1+\epsilon}\right), \qquad (2)\end{aligned}$$

where $\varepsilon > 0$ is arbitrarily small and the constant in the O-term depends on r, ε and f_1. By Lemma 2 we obtain that

$$\begin{aligned}\tilde{D}_2 = &\frac{2A_r r^4}{\log K} \int_1^K \frac{P_1(\log x_1)}{x_1} \int_1^{\frac{K}{x_1}} f_1\left(\frac{\log K/x_1 x}{\log K}\right) \\ &\times \tilde{f}_1\left(\frac{\log K/x_1 x}{\log K}\right)(\log x)^{r^2-1}\frac{dx}{x}dx_1 + O\left((\log T)^{r^2-1+\varepsilon}\right),\end{aligned}$$

where $P_1(y) = \frac{P(y)}{y}$. By the variable changes $u = 1 - \frac{\log x_1}{\log K}$, $v = 1 - \frac{\log x_1 x}{\log K}$, we have

$$\begin{aligned}\tilde{D}_2 &= 2A_r r^4 (\log K)^{r^2} \int_0^1 P_1(1-u) \int_0^u (u-v)^{r^2-1} f_1(v)\tilde{f}_1(v) dv du \\ &\quad + O\left((\log T)^{r^2-1+\varepsilon}\right) \\ &= 2A_r r^4 (\log T)^{r^2} \int_0^1 P_1(1-u) \int_0^u (u-v)^{r^2-1} f_1(v)\tilde{f}_1(v) dv du \\ &\quad + O\left((\log T)^{r^2-1+\varepsilon}\right), \qquad (3)\end{aligned}$$

where the constant in the O-term depends on r, ε and $f_1, \tilde{f}_1$.

We have

$$\begin{aligned}\tilde{D}_3 &= \sum_{k\leq K} \frac{d_r(k)^2}{k} \tilde{f}_1\left(\frac{\log K/k}{\log K}\right)^2 \sum_{p_1 p_2|k} \mu^2(p_1 p_2) P\left(\frac{\log p_1}{\log K}\right) P\left(\frac{\log p_2}{\log K}\right) \\ &\quad + \sum_{k\leq K} \frac{d_r(k)^2}{k} \tilde{f}_1\left(\frac{\log K/k}{\log K}\right)^2 \sum_{p|k} P^2\left(\frac{\log p}{\log K}\right) \\ &= \tilde{D}_{31} + \tilde{D}_{32}.\end{aligned}$$

Again by Lemma 2 we obtain that

$$\tilde{D}_{31} = \frac{A_r r^6}{\log^2 K} \int_1^K \frac{P_1(\log x_1)}{x_1} \int_1^{\frac{K}{x_1}} \frac{P_1(\log x_2)}{x_2} \int_1^{\frac{K}{x_1 x_2}} \tilde{f}_1 \left(\frac{\log K/x_1 x_2 x}{\log K} \right)^2 \times (\log x)^{r^2-1} \frac{dx}{x} dx_2 dx_1 + O\left((\log T)^{r^2-1+\varepsilon} \right),$$

where $P_1(y) = \frac{P(y)}{y}$. We remark that by Lemma 1 we can reduce the number of the repeated integrations in the above expression. By the change of variables $u = 1 - \frac{\log x_1}{\log K}$, $v = 1 - \frac{\log x_2}{\log K}$, $w = 1 - \frac{\log x_1 x_2 x}{\log K}$,

$$\begin{aligned} \tilde{D}_{31} &= A_r r^6 (\log K)^{r^2} \int_0^1 P_1(1-u) \int_{1-u}^1 P_1(1-v) \\ &\quad \times \int_0^{u+v-1} (u+v-w-1)^{r^2-1} \tilde{f}_1(w)^2 dw\, dv\, du \\ &\quad + O\left((\log T)^{r^2-1+\epsilon} \right) \\ &= A_r r^6 (\log T)^{r^2} \int_0^1 P_1(1-u) \int_{1-u}^1 P_1(1-v) \\ &\quad \times \int_0^{u+v-1} (u+v-w-1)^{r^2-1} \tilde{f}_1(w)^2 dw\, dv\, du \\ &\quad + O\left((\log T)^{r^2-1+\varepsilon} \right), \end{aligned} \tag{4}$$

where the constant in the O-term depends on r, ε and $\tilde{f}_1$. Similarly,

$$\begin{aligned} \tilde{D}_{32} &= A_r r^4 (\log K)^{r^2} \int_0^1 P_2(1-u) \int_0^u (u-v)^{r^2-1} \tilde{f}_1(v)^2 dv\, du + O\left((\log T)^{r^2-1+\varepsilon} \right) \\ &= A_r r^4 (\log T)^{r^2} \int_0^1 P_2(1-u) \int_0^u (u-v)^{r^2-1} \tilde{f}_1(v)^2 dv\, du + O\left((\log T)^{r^2-1+\varepsilon} \right), \end{aligned} \tag{5}$$

where $P_2(y) = \frac{P(y)^2}{y}$ and the constant in the O-term depends on r, ε and $\tilde{f}_1$.

We now proceed to evaluation of the numerator in the ratio in (1). If we let

$$N(c) = \sum_{nk \le K} a_k a_{nk} g_c(n) \Lambda(n) n^{-1/2},$$

then

$$\begin{aligned} N(c) = \frac{2}{\pi} \sum_{nk \le K} & \frac{\lambda(k) d_r(k) \lambda(nk) d_r(nk) \Lambda(n)}{kn \log n} \sin\left(\pi c \frac{\log n}{\log T} \right) \\ & \times \Bigg(f_1\left(\frac{\log K/k}{\log K} \right) f_1\left(\frac{\log K/nk}{\log K} \right) \\ & + f_1\left(\frac{\log K/nk}{\log K} \right) \tilde{f}_1\left(\frac{\log K/k}{\log K} \right) \sum_{p_1 | k} P\left(\frac{\log p_1}{\log K} \right) \\ & + f_1\left(\frac{\log K/k}{\log K} \right) \tilde{f}_1\left(\frac{\log K/nk}{\log K} \right) \sum_{p_1 | nk} P\left(\frac{\log p_1}{\log K} \right) \\ & + \tilde{f}_1\left(\frac{\log K/k}{\log K} \right) \tilde{f}_1\left(\frac{\log K/nk}{\log K} \right) \sum_{p_1 | k} P\left(\frac{\log p_1}{\log K} \right) \sum_{q_1 | nk} P\left(\frac{\log q_1}{\log K} \right) \Bigg), \end{aligned}$$

so we can write

$$N(c) = N_1 + N_2 + N_3 + N_4.$$

Using the distribution of $\Lambda(n)$, we obtain

$$N_1 = -\frac{2}{\pi}\sum_{pk\le K}\frac{d_r(k)d_r(pk)}{kp}\sin\left(\pi c\frac{\log p}{\log T}\right)f_1\left(\frac{\log K/k}{\log K}\right)f_1\left(\frac{\log K/pk}{\log K}\right)$$
$$+ O\left((\log T)^{r^2-1}\right)$$
$$= -\frac{2r}{\pi}\sum_{p\le K}\frac{\sin\left(\pi c\frac{\log p}{\log T}\right)}{p}\sum_{k\le K/p}\frac{d_r(k)^2}{k}f_1\left(\frac{\log K/k}{\log K}\right)f_1\left(\frac{\log K/pk}{\log K}\right)$$
$$+ O\left((\log T)^{r^2-1}\right).$$

By Lemma 2 we have

$$N_1 = -\frac{2A_r r^3}{\pi}\sum_{p\le K}\frac{\sin\left(\pi c\frac{\log p}{\log T}\right)}{p}\int_1^{\frac{K}{p}} f_1\left(\frac{\log K/x}{\log K}\right)f_1\left(\frac{\log K/px}{\log K}\right)(\log x)^{r^2-1}\frac{dx}{x}$$
$$+ O\left((\log T)^{r^2-1}\right).$$

From the Mertens theorem

$$\sum_{p\le y}\frac{\log p}{p} = \log y + O(1) \tag{6}$$

and Abel's summation, we get

$$N_1 = -\frac{2A_r r^3}{\pi}\int_1^K \frac{\sin\left(\pi c\frac{\log x_1}{\log T}\right)}{x_1\log x_1}\int_1^{\frac{K}{x_1}} f_1\left(\frac{\log K/x}{\log K}\right)f_1\left(\frac{\log K/xx_1}{\log K}\right)(\log x)^{r^2-1}\frac{dx}{x}dx_1$$
$$+ O\left((\log T)^{r^2-1}\right).$$

Interchanging the order of integration and the names of the variables x and x_1, we find

$$N_1 = -\frac{2A_r r^3}{\pi}\int_1^K f_1\left(\frac{\log K/x_1}{\log K}\right)\frac{(\log x_1)^{r^2-1}}{x_1}\int_1^{\frac{K}{x_1}}\frac{\sin\left(\pi c\frac{\log x}{\log T}\right)}{\log x}f_1\left(\frac{\log K/xx_1}{\log K}\right)\frac{dx}{x}dx_1$$
$$+ O\left((\log T)^{r^2-1}\right).$$

Let $u = 1 - \frac{\log x_1}{\log K}$, $v = \frac{\log x}{\log K}$. Then

$$N_1 = -\frac{2A_r r^3}{\pi}(\log K)^{r^2}\int_0^1 (1-u)^{r^2-1}f_1(u)\int_0^u \frac{\sin\left(\pi cv\frac{\log K}{\log T}\right)}{v}f_1(u-v)\,dv\,du$$
$$+ O\left((\log T)^{r^2-1}\right)$$
$$= -\frac{2A_r r^3}{\pi}(\log T)^{r^2}\int_0^1 (1-u)^{r^2-1}f_1(u)\int_0^u \frac{\sin(\pi cv)}{v}f_1(u-v)\,dv\,du$$
$$+ O\left((\log T)^{r^2-1+\varepsilon}\right), \tag{7}$$

where the constant in the O-term depends on r, ε and f_1.

In N_2 we can replace the product of the summation variables nk by pp_1k_0 to get

$$N_2 = -\frac{2r^3}{\pi}\sum_{p_1\le K}\frac{P\left(\frac{\log p_1}{\log K}\right)}{p_1}\sum_{pk_0\le K/p_1}\frac{\sin\left(\pi c\frac{\log p}{\log T}\right)d_r(k_0)^2}{pk_0}$$
$$\times f_1\left(\frac{\log K/(pp_1k_0)}{\log K}\right)\tilde{f}_1\left(\frac{\log K/(p_1k_0)}{\log K}\right) + O\left((\log T)^{r^2-1}\right).$$

The inner sum $\sum_{pk_0\le K/p_1}$ in the expression above is the sum $\sum_{k_0\le K/p_1}\sum_{p\le K/(p_1k_0)}$. As in the calculation of N_1, we can show that this double sum is

$$A_r r^2 \int_1^{\frac{K}{p_1}} \tilde{f}_1\left(\frac{\log K/(p_1x_2)}{\log K}\right)\frac{(\log x_2)^{r^2-1}}{x_2}$$
$$\times \int_1^{\frac{K}{p_1x_2}} \frac{\sin\left(\pi c\frac{\log x}{\log T}\right)}{\log x} f_1\left(\frac{\log K/(p_1xx_2)}{\log K}\right)\frac{dx}{x}dx_2 + O\left((\log T)^{r^2-1}\right).$$

By (6) we obtain

$$N_2 = -\frac{2A_r r^5}{\pi(\log K)}\int_1^K \frac{P_1\left(\frac{\log x_1}{\log K}\right)}{x_1}\int_1^{\frac{K}{x_1}} \tilde{f}_1\left(\frac{\log K/(x_1x_2)}{\log K}\right)\frac{(\log x_2)^{r^2-1}}{x_2}$$
$$\times \int_1^{\frac{K}{x_1x_2}} \frac{\sin\left(\pi c\frac{\log x}{\log T}\right)}{\log x} f_1\left(\frac{\log K/(xx_1x_2)}{\log K}\right)\frac{dx}{x}dx_2\,dx_1 + O\left((\log T)^{r^2-1}\right).$$

Making the variable changes $u = 1-\frac{\log x_1}{\log K}$, $v = 1-\frac{\log x_1x_2}{\log K}$, $w = \frac{\log x}{\log K}$, we get

$$N_2 = -\frac{2A_r r^5}{\pi}(\log T)^{r^2}\int_0^1 P_1(1-u)\int_0^u (u-v)^{r^2-1}\tilde{f}_1(v)$$
$$\times \int_0^v \frac{\sin(\pi cw)}{w} f_1(v-w)\,dw\,dv\,du + O\left((\log T)^{r^2-1+\varepsilon}\right), \tag{8}$$

where the constant in the O-term depends on r, ε and $f_1, \tilde{f}_1$.

As in N_1 and N_2, the terms with $n = p$ for the primes p give the main contribution to N_3:

$$N_3 = -\frac{2}{\pi}\sum_{pk\le K}\sin\left(\pi c\frac{\log p}{\log T}\right)\frac{d_r(k)d_r(kp)}{kp} f_1\left(\frac{\log K/k}{\log K}\right)\tilde{f}_1\left(\frac{\log K/(pk)}{\log K}\right)$$
$$\times \sum_{p_1|pk} P\left(\frac{\log p_1}{\log K}\right) + O\left((\log T)^{r^2-1}\right).$$

For $(p,k)=1$ it follows that

$$\sum_{p_1|pk} P\left(\frac{\log p_1}{\log K}\right) = \sum_{p_1|k} P\left(\frac{\log p_1}{\log K}\right) + P\left(\frac{\log p}{\log K}\right). \tag{9}$$

Since the contribution of the terms with $(p,k)\ne 1$ in N_3 is $O\left((\log T)^{r^2-1}\right)$, then, according to decomposition (9), we can write

$$N_3 = N_{31} + N_{32} + O\left((\log T)^{r^2-1}\right),$$

where

$$N_{31} = -\frac{2r^3}{\pi}\sum_{p_1\le K}\frac{P\left(\frac{\log p_1}{\log K}\right)}{p_1}\sum_{pk_0\le K/p_1}\frac{\sin\left(\pi c\frac{\log p}{\log T}\right)d_r(k_0)^2}{pk_0}$$
$$\times \tilde{f}_1\left(\frac{\log K/(pp_1k_0)}{\log K}\right) f_1\left(\frac{\log K/(p_1k_0)}{\log K}\right) + O\left((\log T)^{r^2-1}\right)$$

and

$$N_{32} = -\frac{2r}{\pi}\sum_{pk\le K}\frac{\sin\left(\pi c\frac{\log p}{\log T}\right)d_r(k)^2 P\left(\frac{\log p}{\log K}\right)}{pk}\tilde{f}_1\left(\frac{\log K/(pk)}{\log K}\right)$$
$$\times f_1\left(\frac{\log K/k}{\log K}\right) + O\left((\log T)^{r^2-1}\right).$$

As in the calculation of N_2 we get

$$N_{31} = -\frac{2A_r r^5}{\pi}(\log T)^{r^2}\int_0^1 P_1(1-u)\int_0^u (u-v)^{r^2-1} f_1(v)$$
$$\times \int_0^v \frac{\sin(\pi c w)}{w}\tilde{f}_1(v-w)\,dw\,dv\,du + O\left((\log T)^{r^2-1+\epsilon}\right),$$

and as in the calculation of N_1,

$$N_{32} = -\frac{2A_r r^3}{\pi}(\log T)^{r^2}\int_0^1 (1-u)^{r^2-1} f_1(u)\int_0^u \sin(\pi c v)P_1(v)\tilde{f}_1(u-v)\,dv\,du$$
$$+ O\left((\log T)^{r^2-1+\varepsilon}\right),$$

where $P_1(y) = \frac{P(y)}{y}$.

Thus,

$$N_3 = -\frac{2A_r r^5}{\pi}(\log T)^{r^2}\int_0^1 P_1(1-u)\int_0^u (u-v)^{r^2-1} f_1(v)$$
$$\times \int_0^v \frac{\sin(\pi c w)}{w}\tilde{f}_1(v-w)\,dw\,dv\,du$$
$$-\frac{2A_r r^3}{\pi}(\log T)^{r^2}\int_0^1 (1-u)^{r^2-1} f_1(u)\int_0^u \sin(\pi c v)P_1(v)\tilde{f}_1(u-v)\,dv\,du$$
$$+ O\left((\log T)^{r^2-1+\varepsilon}\right), \tag{10}$$

where the constant in the O-term depends on r, ε and $f_1, \tilde{f}_1, P$.

Again, in the sum defining N_4 we can replace the integers $n \geq 2$ with the primes p:

$$N_4 = -\frac{2}{\pi}\sum_{pk\leq K}\sin\left(\pi c\frac{\log p}{\log T}\right)\frac{d_r(k)d_r(kp)}{kp}\tilde{f}_1\left(\frac{\log K/k}{\log K}\right)\tilde{f}_1\left(\frac{\log K/(pk)}{\log K}\right)$$
$$\times \sum_{p_1|k} P\left(\frac{\log p_1}{\log K}\right)\sum_{q_1|pk} P\left(\frac{\log q_1}{\log K}\right) + O\left((\log T)^{r^2-1}\right).$$

For the two innermost sums, if $(k,p)=1$, we have

$$\sum_{p_1|k} P\left(\frac{\log p_1}{\log K}\right)\sum_{q_1|pk} P\left(\frac{\log q_1}{\log K}\right) = \sum_{p_1 q_1|k}\mu^2(p_1 q_1)P\left(\frac{\log p_1}{\log K}\right)P\left(\frac{\log q_1}{\log K}\right)$$
$$+\sum_{p_1|k} P^2\left(\frac{\log p_1}{\log K}\right) + P\left(\frac{\log p}{\log K}\right)\sum_{p_1|k} P\left(\frac{\log p_1}{\log K}\right).$$

According to this decomposition, we write

$$N_4 = N_{41} + N_{42} + N_{43}.$$

As before, by Lemma 2 we find

$$N_{41} = -\frac{2A_r r^7}{\pi(\log K)^2}\int_1^K \frac{P_1\left(\frac{\log x_1}{\log K}\right)}{x_1}\int_1^{\frac{K}{x_1}} \frac{P_1\left(\frac{\log x_2}{\log K}\right)}{x_2}$$
$$\times \int_1^{\frac{K}{x_1 x_2}} \tilde{f}_1\left(\frac{\log K/(x_1 x_2 x_3)}{\log K}\right)\frac{(\log x_3)^{r^2-1}}{x_3}$$
$$\times \int_1^{\frac{K}{x_1 x_2 x_3}} \frac{\sin\left(\pi c\frac{\log x}{\log T}\right)}{\log x}\tilde{f}_1\left(\frac{\log K/(x x_1 x_2 x_3)}{\log K}\right)\frac{dx}{x}dx_3\,dx_2\,dx_1$$
$$+ O\left((\log T)^{r^2-1}\right).$$

Making the variable changes $u = 1 - \frac{\log x_1}{\log K}$, $v = 1 - \frac{\log x_2}{\log K}$, $w = 1 - \frac{\log x_1 x_2 x_3}{\log K}$, $z = \frac{\log x}{\log K}$, we get

$$N_{41} = -\frac{2A_r r^7}{\pi}(\log T)^{r^2} \int_0^1 P_1(1-u) \int_{1-u}^1 P_1(1-v) \times \int_0^{u+v-1} (u+v-w-1)^{r^2-1} \tilde{f}_1(w) \int_0^w \frac{\sin(\pi c z)}{z} \tilde{f}_1(w-z)\, dz\, dw\, dv\, du + O\left((\log T)^{r^2-1+\varepsilon}\right), \tag{11}$$

where the constant in the O-term depends on r, ε and $\tilde{f}_1$, P.

Next,

$$N_{42} = -\frac{2A_r r^5}{\pi(\log K)} \int_1^K \frac{P_2\left(\frac{\log x_1}{\log K}\right)}{x_1} \int_1^{\frac{K}{x_1}} \tilde{f}_1\left(\frac{\log K/(x_1 x_2)}{\log K}\right) \frac{(\log x_2)^{r^2-1}}{x_2} \times \int_1^{\frac{K}{x_1 x_2}} \frac{\sin\left(\pi c \frac{\log x}{\log T}\right)}{\log x} \tilde{f}_1\left(\frac{\log K/(x x_1 x_2)}{\log K}\right) \frac{dx}{x} dx_2\, dx_1 + O\left((\log T)^{r^2-1}\right).$$

By the variable changes $u = 1 - \frac{\log x_1}{\log K}$, $v = 1 - \frac{\log x_1 x_2}{\log K}$, $w = \frac{\log x}{\log K}$, we get

$$N_{42} = -\frac{2A_r r^5}{\pi}(\log T)^{r^2} \int_0^1 P_2(1-u) \int_0^u (u-v)^{r^2-1} \tilde{f}_1(v) \times \int_0^v \frac{\sin(\pi c w)}{w} \tilde{f}_1(v-w)\, dw\, dv\, du + O\left((\log T)^{r^2-1+\varepsilon}\right), \tag{12}$$

where the constant in the O-term depends on r, ϵ and $\tilde{f}_1$, P.

Finally,

$$N_{43} = -\frac{2A_r r^5}{\pi(\log K)^2} \int_1^K \frac{P_1\left(\frac{\log x_1}{\log K}\right)}{x_1} \int_1^{\frac{K}{x_1}} \tilde{f}_1\left(\frac{\log K/(x_1 x_2)}{\log K}\right) \frac{(\log x_2)^{r^2-1}}{x_2} \times \int_1^{\frac{K}{x_1 x_2}} \sin\left(\pi c \frac{\log x}{\log T}\right) P_1\left(\frac{\log x}{\log K}\right) \tilde{f}_1\left(\frac{\log K/(x x_1 x_2)}{\log K}\right) \frac{dx}{x} dx_2\, dx_1 + O\left((\log T)^{r^2-1}\right).$$

By the variable changes $u = 1 - \frac{\log x_1}{\log K}$, $v = 1 - \frac{\log x_1 x_2}{\log K}$, $w = \frac{\log x}{\log K}$, we get

$$N_{43} = -\frac{2A_r r^5}{\pi}(\log T)^{r^2} \int_0^1 P_1(1-u) \int_0^u (u-v)^{r^2-1} \tilde{f}_1(v) \times \int_0^v \sin(\pi c w) P_1(w) \tilde{f}_1(v-w)\, dw\, dv\, du + O\left((\log T)^{r^2-1+\epsilon}\right), \tag{13}$$

where the constant in the O-term depends on r, ϵ and $\tilde{f}_1$, P.

Using D_i, N_i given by (2)–(13) we can evaluate (note the sign change in comparison with $h(c)$ for μ in [5])

$$h(c) = c - \frac{N_1 + N_2 + N_3 + N_4}{D_1 + D_2 + D_3}.$$

The results of our numerical calculations are summarized in Tables 1 and 2. To find the coefficients of the numerically optimal polynomials, we perform one iteration of Newton's method for multidimensional optimization, using in the initial vector the coefficients

Table 1 Numerically optimal polynomials in the coefficients $\{a_k\}$, for which $h(c) > 1$

Degrees			Value of c	Value of r	Polynomials		
f_1	$\tilde{f}_1$	P			f_1	$\tilde{f}_1$	P
3	1	2	0.515398	1.18	$1.95 + 1.47x - 1.07x^2 - 0.29x^3$	$-0.7 - 1.92x$	x^2
3	1	3	0.515397	1.18	$1.655 + 1.25x - 0.886x^2 - 0.25x^3$	$-0.57 - 1.6x$	$x^2 + 0.036x^3$
6	2	3	0.515396	1.18	$1.78 + 1.017x + 0.2x^2 - 1.56x^3 + 0.45x^4 - 0.06x^5 + 0.05x^6$	$-0.629 - 0.88x - 1.799x^2$	$x^2 + 0.083x^3$

Table 2 Numerically optimal polynomials in the coefficients $\{a_k/\lambda(k)\}$, for which $h(c) < 1$ [(see (1))]

Degrees			Value of c	Value of r	Polynomials		
f_1	$\tilde{f}_1$	P			f_1	$\tilde{f}_1$	P
5	3	2	2.73272	2.6	$1.02 + 10.96x + 9.29x^2 - 22.3x^3 - 26.18x^4 + 34.45x^5$	$-4.56 - 63.02x - 42.72x^2 - 34.45x^3$	x^2
6	3	3	2.7328	2.6	$1.08 + 11.79x + 8.61x^2 - 21.95x^3 - 24.58x^4 + 27.79x^5 + 5.03x^6$	$-4.89 - 69.11x - 36.7x^2 - 51.28x^3$	$x^2 - 0.044x^3$

found by Feng and Wu [5]. To obtain the bound for λ in Theorem 1, we use the relaxed Newton's method with the step size multiplier equal to 0.01.

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