Research in Number Theory

RESEARCH Open Access

Congruences for generalized Apéry numbers and Gaussian hypergeometric series

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Abstract

For positive integers f_1, f_2, m, l, we define a generalization of Apéry numbers $A(f_1, f_2, m, l, \lambda)$ given by

$$A(f_1, f_2, m, l, \lambda) := \sum_{j=0}^{f_2} \binom{f_1+j}{j}^m \binom{f_2}{j}^l \lambda^j.$$

In this article, we deduce congruence relations satisfied by these generalized Apéry numbers extending results of (Coster in Supercongruences, Ph.D. thesis, Universiteit Leiden, 1988). We find expressions of $A(f_1, f_2, m, l, \lambda)$ in terms of Gaussian hypergeometric series and evaluate some new supercongruences similar to Beukers' supercongruences.

Keywords: Apéry numbers, Gaussian hypergeometric series, Supercongruences

Mathematics Subject Classification: 11A07, 33C20, 12E20

1 Introduction and statement of results

In [3], R. Apéry introduced the numbers a_n and b_n commonly known as Apéry numbers defined by the explicit formulas

$$a_n := \sum_{k=0}^{n} \binom{n+j}{j}^2 \binom{n}{j}^2 \quad \text{and} \quad b_n := \sum_{k=0}^{n} \binom{n+j}{j} \binom{n}{j}^2$$

in his irrationality proof of $\zeta(3)$ and $\zeta(2)$, respectively. Since their appearance, mathematicians started looking more closely at these numbers and found many interesting properties satisfied by them. Chowla et al. [11] were first to consider congruence properties for the Apéry numbers. They proved many interesting congruence relations for the Apéry numbers and listed a few conjectures relating them. For example, they proved that for primes $p \geq 5$,

$$a_p \equiv a_1 \pmod{p^3}.$$

Gessel [13] generalized this to the stronger result

$$a_{np} \equiv a_n \pmod{p^3}. \tag{1.1}$$

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In a recent paper [10], Chan, Cooper, and Sica investigated certain sequences of integers $\{f_n\}_{n=1}^{\infty}$ that satisfy relations similar to (1.1). The identification of a_n as the coefficients of certain power series motivates them to obtain Apéry-like sequences $\{f_n\}_{n=1}^{\infty}$ satisfying congruences similar to (1.1).

One main purpose of this paper is to find similar congruences for the numbers $A(f_1, f_2, m, l, \lambda)$ given by the formula

$$A(f_1, f_2, m, l, \lambda) := \sum_{j=0}^{f_2} \binom{f_1 + j}{j}^m \binom{f_2}{j}^l \lambda^j,$$

where $f_1, f_2, m, l \in \mathbb{N}$. Note that this family of sequences includes the Apéry numbers. In his PhD thesis [12], Coster studied the numbers $A(f_1, f_1, l, m, \lambda)$ and proved that

$$A(f_1 p^r, f_1 p^r, m, l, \lambda) \equiv A(f_1 p^{r-1}, f_1 p^{r-1}, m, l, \lambda) \pmod{p^{3r}}$$

if $p \geq 5$ and $\lambda = \pm 1$. Our main result extends the result of Coster and proves the following congruence for the numbers $A(f_1, f_2, m, l, \pm 1)$.

Theorem 1.1 *Let $\lambda = \pm 1$. For primes $p > 3$ and integers $l \geq 2$, $r \geq 1$, we have*

$$A(f_1 p^r, f_2 p^r, m, l, \lambda) \equiv A(f_1 p^{r-1}, f_2 p^{r-1}, m, l, \lambda) \pmod{p^{3r}}.$$

Let p be an odd prime, and $\mathbb{F}_p$ denote the finite field with p elements. A multiplicative character $\chi : \mathbb{F}_p^{\times} \to \mathbb{C}$ is a group homomorphism. The set $\widehat{\mathbb{F}_p^{\times}}$ of all multiplicative characters on $\mathbb{F}_p^{\times}$ forms a cyclic group under multiplication of characters. Extend each character $\chi \in \widehat{\mathbb{F}_p^{\times}}$ to all of $\mathbb{F}_p$ by setting $\chi(0) := 0$. The binomial coefficient $\binom{A}{B}$ is defined by

$$\binom{A}{B} := \frac{B(-1)}{q} J(A, \overline{B}) = \frac{B(-1)}{p} \sum_{x \in \mathbb{F}_p} A(x) \overline{B}(1 - x),$$

where $J(A, B)$ denotes the usual Jacobi sum and $\overline{B}$ is the inverse of B. The following property of binomial coefficients is known from [14]

$$\binom{A}{B} = \binom{B\overline{A}}{B} B(-1). \tag{1.2}$$

With this notation, for characters $A_0, A_1, \ldots, A_r$ and $B_1, B_2, \ldots, B_r$ of $\mathbb{F}_p$, Greene [14] defined the Gaussian hypergeometric series ${}_{r+1}F_r\left(\begin{matrix} A_0, & A_1, & \ldots, & A_r \\ & B_1, & \ldots, & B_r \end{matrix} \mid x\right)$ over $\mathbb{F}_p$ as

$${}_{r+1}F_r\left(\begin{matrix} A_0, & A_1, & \ldots, & A_r \\ & B_1, & \ldots, & B_r \end{matrix} \mid x\right) := \frac{p}{p-1} \sum_{\chi} \binom{A_0\chi}{\chi} \binom{A_1\chi}{B_1\chi} \cdots \binom{A_r\chi}{B_r\chi} \chi(x),$$

where the sum is over all characters χ of $\mathbb{F}_p$.

For an odd prime p, Beukers [9] and Stienstra-Beukers [25] proved many interesting congruence properties satisfied by Apéry numbers, and they respectively made the following two conjectures concerning the Apéry numbers and the coefficients of two modular forms:

$$a_{\frac{p-1}{2}} \equiv \alpha(p) \pmod{p^2}, \tag{1.3}$$

$$b_{\frac{p-1}{2}} \equiv \beta(p) \pmod{p^2}, \tag{1.4}$$

where

$$\sum_{n=0}^{\infty} \alpha(n)q^n := q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4,$$
$$\sum_{n=0}^{\infty} \beta(n)q^n := q\prod_{n=1}^{\infty}(1-q^{4n})^6.$$

For $p \nmid a_{\frac{p-1}{2}}$, Ishikawa [17] gave a proof (1.3). In [1], Ahlgren and Ono proved (1.3) by relating the Gaussian hypergeometric series ${}_4F_3(1)$ to $\alpha(p)$ via the modularity of a certain Calabi-Yau threefold. Moreover, they defined the generalized Apéry numbers $A(n, n, l, k, 1)$ and found certain Beukers-like congruences for these generalized Apéry numbers exploring connections between Gaussian hypergeometric series, elliptic curves and the Dedekind eta function. Following the technique of [1], Ahlgren [2] studied ${}_3F_2(\lambda)$ Gaussian hypergeometric series using tools from p-adic analysis and deduced (1.4), which was already proved by vanHamme [26] for $p \equiv 1 \pmod 4$ and by Ishikawa [16] in general. Further, he evaluated some new supercongruences similar to (1.4) by inserting Beukers' supercongruences into a larger framework.

One of our main aims in this paper is to investigate similar phenomena for the generalized Apéry numbers $A(m, n, l, k, \lambda)$ and Gaussian hypergeometric series ${}_{r+1}F_r(\lambda)$. In [20], Koike found an expression of $A\left(\frac{p-1}{2}, \frac{p-1}{2}, m, l, 1\right)$ in terms of Gaussian hypergeometric series with quadratic and trivial characters as parameters. Following this, Ono [22] deduced a similar congruence relation for $A\left(\frac{p-1}{2}, \frac{p-1}{2}, m, l, \lambda\right)$. In the following theorem, we extend these results of [20,22] and evaluate an expression of $A\left(\frac{p-1}{r}, \frac{p-1}{s}, m, l, \lambda\right)$ in terms of Gaussian hypergeometric series.

Theorem 1.2 *Let p be an odd prime such that $p \equiv 1 \pmod{lcm(r, s)}$. If $w = m + l$ and $0 < a < r$, $0 < b < s$, then*

$$A\left(\frac{a(p-1)}{r}, \frac{b(p-1)}{s}, m, l, \lambda\right) \equiv (-p)^{w-1}\, {}_wF_{w-1}\left(\begin{matrix}\psi^a, \psi^a, \ldots, \psi^a, \overline{\xi}^b, \overline{\xi}^b, \ldots, \overline{\xi}^b \\ \epsilon, \ldots, \epsilon, \epsilon, \epsilon, \cdots, \epsilon\end{matrix} \mid (-1)^l\lambda\right) \pmod p,$$

where the character ψ^a of order r appears m times and the character ξ^b of order s appears l times.

As a consequence of the above theorem, we evaluate the following congruences for the generalized Apéry numbers.

Theorem 1.3 *Let $p \equiv 1 \pmod 4$ be a prime and $d = \frac{p-1}{4}$, then*

$$(i)\ A\left(d, d, 1, 1, -\frac{4}{3}\right) \equiv \chi_4(-1)A\left(d, d, 1, 1, \frac{1}{3}\right) \equiv \begin{cases} 0 \pmod p, & \text{if } p \equiv 2 \pmod 3; \\ -2a_3\chi_4(-36) \pmod p, & \text{if } p \equiv 1 \pmod 3, \end{cases}$$

$$(ii)\ A(d,d,1,1,-4)\equiv \chi_4(4)A\left(d,d^3,1,1,-\frac{1}{4}\right)$$
$$\equiv \begin{cases} 0 \pmod p, & \text{if } p\equiv 2 \pmod 3;\\ -2a_3\chi_4(12) \pmod p, & \text{if } p\equiv 1 \pmod 3,\end{cases}$$

where $p=a_3^2+3b_3^2$ *and* $a_3\equiv -1 \pmod 3$, *when* $p\equiv 1 \pmod 3$.

We now move our attention to supercongruence relations satisfied by the generalized Apéry numbers $A\left(\frac{r-1}{r}(p-1),\frac{p-1}{r},m,m(r-1),\lambda\right)$. For odd prime p, we define

$$B(f_1,f_2,m,l,\lambda):=\sum_{j=0}^{f_2}\binom{f_1+j}{j}^m\binom{f_2}{j}^l\lambda^j\left\{1+\frac{m(p-1)}{f_2}j\left\{H_{\frac{pf_2-f_1}{p-1}+j}-H_j\right\}\right\},$$

where $H_0:=0$ and $H_n:=1+\frac{1}{2}+\ldots+\frac{1}{n}$ for $n\in\mathbb{N}$. In the following theorem, we express Gaussian hypergeometric series in terms of the quantities $A(f_1,f_2,m,l,\lambda)$ and $B(f_1,f_2,m,l,\lambda)$.

Theorem 1.4 *Let* $p\equiv 1 \pmod r$ *and* $T\in\widehat{\mathbb{F}_q^\times}$ *be a generator of the character group, then*

$$(-p)^{mr-1}{}_{mr}F_{mr-1}\left(\begin{matrix} T^{\frac{p-1}{r}}, & T^{\frac{p-1}{r}}, & \ldots, & T^{\frac{p-1}{r}}\\ & \epsilon, & \ldots, & \epsilon\end{matrix} \mid (-1)^{m(r-1)}\lambda\right)$$
$$\equiv A\left(\frac{r-1}{r}(p-1),\frac{p-1}{r},m,m(r-1),\lambda^p\right)$$
$$+pB\left(\frac{r-1}{r}(p-1),\frac{p-1}{r},m,m(r-1),\lambda\right) \pmod{p^2}.$$

Remark 1.5 In [1], Ahlgren and Ono obtained a particular case of Theorem 1.4 with $r=2, m=2$. Moreover, they deduced the Beukers' conjecture (1.3) by proving the combinatorial identity $B\left(\frac{p-1}{2},\frac{p-1}{2},2,2,1\right)\equiv 0 \pmod p$.

Remark 1.6 For $r=2$ and $m\in\mathbb{N}$, Osburn and Schneider [23] found a modulo p^3 version of Theorem 1.4 extending results of [1,2]. In particular, they deduced a supercongruence for the Legendre symbol $\left(\frac{-1}{p}\right)$, which generalized a supercongruence conjecture of Rodriguez-Villegas proved by Mortenson [21].

For $\lambda\in\mathbb{Q}\setminus\{0,1\}$, consider the family of elliptic curves

$${}_2E_1(\lambda): y^2=x(x-1)(x-\lambda)$$

and denote by ${}_2a_1(p,\lambda)$ the trace of the Frobenius endomorphism of ${}_2E_1(\lambda)$ over $\mathbb{F}_p$. Then the associated Hasse-Weil L-function of ${}_2F_1(\lambda)$ is given by $L({}_2E_1(\lambda),s)=\sum_{n=1}^{\infty}\frac{{}_2a_1(n,\lambda)}{n^s}$. Koike [20] proved that if p is an odd prime for which $\mathrm{ord}_p(\lambda(\lambda-1))=0$, then

$${}_2a_1(p,\lambda)=-p\phi(-1){}_2F_1\left(\begin{matrix}\phi, & \phi\\ & \epsilon\end{matrix}\mid\lambda\right).$$

This, together with Theorem 1.4 yields

Corollary 1.7 *If p is an odd prime and* $\lambda\in\mathbb{Q}\setminus\{0,1\}$ *such that* $\mathrm{ord}_p(\lambda(\lambda-1))=0$, *then*

$${}_2a_1(p,\lambda)\equiv\phi(-1)A\left(\frac{p-1}{2},\frac{p-1}{2},1,1,-\lambda^p\right)$$

$$+\,\phi(-1)pB\left(\frac{p-1}{2},\frac{p-1}{2},1,1,-\lambda\right) \pmod{p^2}.$$

A particular case of [5, Theorem. 3.2] states that if $p \equiv 1 \pmod 4$, then the trace of Frobenius map on the elliptic curve $E(\lambda): y^2 = x^3 + \lambda x^2 + x$ over $\mathbb{F}_p$ is given by

$$a(p,\lambda) = -p\phi(2\lambda)\chi_4(-1)_2F_1\left(\begin{matrix}\chi_4, \chi_4^3 \\ \epsilon\end{matrix} \mid \frac{4}{\lambda^2}\right).$$

With a change of variables in [18, (4.2)], we obtain

$${}_2F_1\left(\begin{matrix}\phi, \phi \\ \varepsilon\end{matrix} \mid \frac{4}{2-\lambda}\right) = \phi(2\lambda(\lambda-2))\chi_4(-1)_2F_1\left(\begin{matrix}\chi_4, \chi_4^3 \\ \varepsilon\end{matrix} \mid \frac{4}{\lambda^2}\right).$$

As a result, we deduce

$$a(p,\lambda) = -p\phi(\lambda-2)_2F_1\left(\begin{matrix}\phi, \phi \\ \epsilon\end{matrix} \mid \frac{4}{2-\lambda}\right).$$

Therefore, we have the following immediate consequence of Theorem 1.4.

Corollary 1.8 *Let p be an odd prime and $\lambda \in \mathbb{Q}^\times$ such that $ord_p(\lambda^2-4)=0$, then*

$$a(p,\lambda) \equiv \phi(\lambda-2)A\left(\frac{p-1}{2},\frac{p-1}{2},1,1,\left(\frac{4}{\lambda-2}\right)^p\right)$$
$$+\,\phi(\lambda-2)pB\left(\frac{p-1}{2},\frac{p-1}{2},1,1,\frac{4}{\lambda-2}\right) \pmod{p^2}.$$

The organization of this paper is as follows. In Section 2, we prove Theorem 1.1 using the techniques of [13] and [24]. We prove Theorem 1.2 in Section 3, and deduce the congruence relations for $A(\frac{p-1}{r},\frac{p-1}{s},m,l,\lambda)$ stated in Theorem 1.3. In Section 4, we recall necessary properties of Jacobi sums and p-adic Gamma function required to prove Theorem 1.4 in Section 5. Finally, in the last section we give examples of certain supercongruences similar to Beukers' supercongruences as given in [2, Section 5].

2 Proof of Theorem 1.1

We recall the following result from [24].

Lemma 2.1 [24, Lemma 2.2] *Let p be a prime and n an integer such that $n \not\equiv 0 \pmod{p-1}$. Then, for all integers $r \geq 0$,*

$$\sum_{k=1;p\nmid k}^{p^r-1} k^n \equiv 0 \pmod{p^r}.$$

If, additionally, n is even, then, for primes $p \geq 5$,

$$\sum_{k=1;p\nmid k}^{(p^r-1)/2} \frac{1}{k^n} \equiv 0 \pmod{p^r}.$$

In addition, we prove the following lemma.

Lemma 2.2 *Let $p \geq 5$ be a prime and n be an even integer such that $n \not\equiv 0 \pmod{p-1}$. Then for all integers $r > 0$,*

$$\sum_{k=1;p\nmid k}^{p^r-1} \frac{(-1)^k}{k^n} \equiv 0 \pmod{p^r}.$$

Proof If $p \nmid k$, then $p \nmid (p^r - k)$ for any $r > 0$. Therefore,

$$\sum_{k=1;p\nmid k}^{p^r-1} \frac{(-1)^k}{k^n} = \sum_{k=1;p\nmid k}^{\frac{p^r-1}{2}} \left\{ \frac{(-1)^k}{k^n} + \frac{(-1)^{p^r-k}}{(p^r-k)^n} \right\} \equiv \sum_{k=1}^{\frac{p^r-1}{2}} (-1)^k \left\{ \frac{1}{k^n} - \frac{1}{(-k)^n} \right\} \pmod{p^r}.$$

Since n is even, we complete the proof of the lemma. □

Lemma 2.3 *For prime p and integers $f_1, f_2, k \geq 0, r \geq 1, A \geq 0, B \geq 0$, we have*

$$\binom{f_2 p^r - 1}{k}^A \binom{f_1 p^r + k}{k}^B \equiv (-1)^{\left(k + \left[\frac{k}{p}\right]\right)A} \binom{f_2 p^{r-1} - 1}{\left[\frac{k}{p}\right]}^A \binom{f_1 p^{r-1} + \left[\frac{k}{p}\right]}{\left[\frac{k}{p}\right]}^B \pmod{p^r}.$$

Proof The proof of the lemma follows immediately from [24, Lemma 2.5]. □

Proof of Theorem 1.1 Let $\lambda = \pm 1$. Splitting into two sums, we obtain

$$\begin{aligned} A(f_1 p^r, f_2 p^r, m, l, \lambda) &= \sum_{j=0}^{f_2 p^r} \binom{f_1 p^r + j}{j}^m \binom{f_2 p^r}{j}^l \lambda^j \\ &= \sum_{j=0;p\nmid j}^{f_2 p^r} \binom{f_1 p^r + j}{j}^m \binom{f_2 p^r}{j}^l \lambda^j + \sum_{j=0;p\mid j}^{f_2 p^r} \binom{f_1 p^r + j}{j}^m \binom{f_2 p^r}{j}^l \lambda^j \\ &:= S_1 + S_2. \end{aligned} \tag{2.1}$$

If $p \mid j$, then $j = kp^s$ for some $k, s \geq 0$ and $p \nmid k$. Therefore,

$$S_2 = \sum_{k=0}^{f_2} \sum_{s=1}^{r} \binom{f_1 p^r + kp^s}{kp^s}^m \binom{f_2 p^r}{kp^s}^l \lambda^k.$$

Clearly $s \leq r$, and hence using Jacobsthal's congruences [24, Lemma 2.1]

$$\binom{f_2 p^r}{kp^s} \equiv \binom{f_2 p^{r-1}}{kp^{s-1}} \pmod{p^{r+2s}}$$

and

$$\binom{f_1 p^r + kp^s}{kp^s} \equiv \binom{f_1 p^{r-1} + kp^{s-1}}{kp^{s-1}} \pmod{p^{r+2s}}$$

we deduce that

$$\begin{aligned} S_2 &\equiv \sum_{k=0}^{f_2} \sum_{s=1}^{r} \binom{f_1 p^{r-1} + kp^{s-1}}{kp^{s-1}}^m \binom{f_2 p^{r-1}}{kp^{s-1}}^l \lambda^{kp^{r+2s-1}} \pmod{p^{r+2s}} \\ &\equiv \sum_{k=0}^{f_2} \sum_{s=1}^{r} \binom{f_1 p^{r-1} + kp^{s-1}}{kp^{s-1}}^m \binom{f_2 p^{r-1}}{kp^{s-1}}^l \lambda^k \pmod{p^{r+2s}}. \end{aligned} \tag{2.2}$$

For $p \nmid k$, we have

$$\binom{f_2p^r}{kp^s}^l \equiv \binom{f_2p^{r-1}}{kp^{s-1}}^l \equiv 0 \pmod{p^{l(r-s)}}.$$

In particular,

$$\sum_{k=0}^{f_2}\sum_{s=1}^{r}\binom{f_1p^{r-1}+kp^{s-1}}{kp^{s-1}}^m\binom{f_2p^{r-1}}{kp^{s-1}}^l\lambda^k \equiv \sum_{k=0}^{f_2}\sum_{s=1}^{r}\binom{f_1p^r+kp^s}{kp^s}^m\binom{f_2p^r}{kp^s}^l\lambda^k \equiv 0 \pmod{p^{2(r-s)}}. \tag{2.3}$$

Since $r+2s+2(r-s)=3r$, (2.2) and (2.3) give

$$S_2 = \sum_{j=0}^{f_2p^{r-1}}\binom{f_1p^{r-1}+j}{j}^m\binom{f_2p^{r-1}}{j}^l\lambda^j \equiv A(f_1p^{r-1}, f_2p^{r-1}, m, l, \lambda) \pmod{p^{3r}}. \tag{2.4}$$

Again if $p \nmid j$, then

$$ord_p\binom{f_2p^r}{j}^l = ord_p\binom{f_2p^r}{jp^0}^l \geq lr.$$

In particular,

$$S_1 \equiv 0 \pmod{p^{3r}}$$

for $l \geq 3$, and hence the proof follows from (2.1) because of (2.4) when $l \geq 3$. Thus we are left to prove the case when $l=2$. We now follow the approach of [24]. For $l=2$, we need to show that

$$S_1 \equiv \sum_{j=0;p\nmid j}^{f_2p^r}\binom{f_1p^r+j}{j}^m\binom{f_2p^r}{j}^2\lambda^j \equiv (f_2p^r)^2\sum_{j=0;p\nmid j}^{f_2p^r}\frac{1}{j^2}\binom{f_1p^r+j}{j}^m\binom{f_2p^r-1}{j-1}^2\lambda^j \equiv 0 \pmod{p^{3r}},$$

which is equivalent to

$$\sum_{j=0;p\nmid j}^{f_2p^r}\frac{1}{j^2}\binom{f_1p^r+j}{j}^m\binom{f_2p^r-1}{j-1}^2\lambda^j \equiv 0 \pmod{p^r}. \tag{2.5}$$

Using Lemma 2.3, we obtain

$$\sum_{j=0;p\nmid j}^{f_2p^r}\frac{1}{j^2}\binom{f_1p^r+j}{j}^m\binom{f_2p^r-1}{j-1}^2\lambda^j \equiv \sum_{j=0;p\nmid j}^{f_2p^{r-1}}\frac{1}{j^2}\binom{f_1p^{r-1}+\left[\frac{j}{p}\right]}{\left[\frac{j}{p}\right]}^m\binom{f_2p^{r-1}-1}{\left[\frac{j}{p}\right]}^2\lambda^j \pmod{p^r} \tag{2.6}$$

because of the fact that $\left[\frac{k-1}{p}\right]=\left[\frac{k}{p}\right]$ when $p \nmid k$. In view of (2.5) and (2.6), it is enough to show that

$$\begin{aligned}
&\sum_{j=0;p\nmid j}^{f_2p^r} \frac{1}{j^2}\binom{f_1p^r+j}{j}^m\binom{f_2p^r-1}{j}^2\lambda^j \\
&\quad\equiv \sum_{j=0;p\nmid j}^{f_2p^{r-s}} \frac{1}{j^2}\binom{f_1p^{r-s}+\left[\frac{j}{p^s}\right]}{\left[\frac{j}{p^s}\right]}^m\binom{f_2p^{r-s}-1}{[\frac{j}{p^s}]}^2\lambda^j \pmod{p^r} \qquad (2.7)
\end{aligned}$$

for $s=0,1,\ldots,r$. The case $s=0$ is trivial, whereas Lemma 2.3 proves it for $s=1$. Let $\{j{:}p^s\}:=j-p^s[\frac{j}{p^s}]$, then

$$\begin{aligned}
&\sum_{j=0;p\nmid j}^{f_2p^{r-s}} \frac{1}{j^2}\binom{f_1p^{r-s}+\left[\frac{j}{p^s}\right]}{\left[\frac{j}{n^s}\right]}^m\binom{f_2p^{r-s}-1}{[\frac{j}{p^s}]}^2\lambda^j \\
&\quad=\sum_n \binom{f_1p^{r-s}+n}{n}^m\binom{f_2p^{r-s}-1}{n}^2 \sum_{j=0;p\nmid j;\left[\frac{j}{p^s}\right]=n}^{f_2p^{r-s}} \frac{\lambda^j}{j^2} \qquad (2.8)\\
&\quad=\sum_n \binom{f_1p^{r-s}+n}{n}^m\binom{f_2p^{r-s}-1}{n}^2 \sum_{j=0;p\nmid j;\left[\frac{j}{p^s}\right]=n;\{j:p^s\}<p^s/2}^{f_2p^{r-s}} \frac{\lambda^j}{j^2}\\
&\qquad+\sum_n \binom{f_1p^{r-s}+n}{n}^m\binom{f_2p^{r-s}-1}{n}^2 \sum_{j=0;p\nmid j;\left[\frac{j}{p^s}\right]=n;\{j:p^s\}>p^s/2}^{f_2p^{r-s}} \frac{\lambda^j}{j^2}. \qquad (2.9)
\end{aligned}$$

It is clear from Lemma 2.2 that p^s divides the inner sum of (2.8) for $\lambda=-1$, and Lemma 2.1 implies that p^s divides each inner sum of (2.9) for $\lambda=1$. For $s<r$, we use Lemma 2.3 in the above expression to obtain

$$\begin{aligned}
&\sum_{j=0;p\nmid j}^{f_2p^{r-s}} \frac{1}{j^2}\binom{f_1p^{r-s}+\left[\frac{j}{p^s}\right]}{\left[\frac{j}{p^s}\right]}^m\binom{f_2p^{r-s}-1}{\left[\frac{j}{p^s}\right]}^2\lambda^j \\
&\quad\equiv\sum_n \binom{f_1p^{r-s-1}+\left[\frac{n}{p}\right]}{\left[\frac{n}{p}\right]}^m\binom{f_2p^{r-s-1}-1}{\left[\frac{n}{p}\right]}^2 \sum_{j=0;p\nmid j;\left[\frac{j}{p^s}\right]=n;\{j:p^s\}<p^s/2}^{f_2p^{r-s}} \frac{\lambda^j}{j^2}\\
&\qquad+\sum_n \binom{f_1p^{r-s-1}+\left[\frac{n}{p}\right]}{\left[\frac{n}{p}\right]}^m\binom{f_2p^{r-s-1}-1}{\left[\frac{n}{p}\right]}^2 \sum_{j=0;p\nmid j;\left[\frac{j}{p^s}\right]=n;\{j:p^s\}>p^s/2}^{f_2p^{r-s}} \frac{\lambda^j}{j^2} \pmod{p^r}\\
&\quad\equiv\sum_{j=0;p\nmid j}^{f_2p^{r-s-1}} \frac{1}{j^2}\binom{f_1p^{r-s-1}+\left[\frac{j}{p^{s+1}}\right]}{\left[\frac{j}{p^{s+1}}\right]}^m\binom{f_2p^{r-s-1}-1}{\left[\frac{j}{p^{s+1}}\right]}^2\lambda^j \pmod{p^r}
\end{aligned}$$

Induction on s completes the proof of (2.7). Moreover, (2.8), (2.9) and (2.7) together imply that

$$\sum_{j=0;p\nmid j}^{f_2p^r} \frac{1}{j^2}\binom{f_1p^r+j}{j}^m\binom{f_2p^r-1}{j}^2\lambda^j \equiv 0 \pmod{p^r},$$

which completes the proof of the theorem. □

3 Proof of Theorem 1.2 and 1.3

Proof of Theorem 1.2 Let ω denote the Teichmüller character defined as $\omega(x) \equiv x \pmod p$ for $x = 0, 1, \ldots, p-1$. Using [20, Lemma 1], we obtain

$$\begin{aligned} A\left(\frac{a(p-1)}{r}, \frac{b(p-1)}{s}, m, l, \lambda\right) &= \sum_{j=0}^{\frac{b(p-1)}{s}} \binom{\frac{a(p-1)}{r}+j}{j}^m \binom{\frac{b(p-1)}{s}}{j}^l \lambda^j \\ &\equiv \left(\frac{p}{p-1}\right)^w \sum_{j=0}^{\frac{b(p-1)}{s}} \binom{\omega^{\frac{a(p-1)}{r}}\omega^j}{\omega^j}^m \binom{\omega^{\frac{b(p-1)}{s}}}{\omega^j}^l \omega^j(\lambda) \pmod p \\ &\equiv \left(\frac{p}{p-1}\right)^w \sum_{\chi} \binom{\psi^a\chi}{\chi}^m \binom{\xi^b\chi}{\chi}^l \chi(\lambda) \pmod p, \end{aligned}$$

where ψ and ξ are characters of order r and s, respectively. Using (1.2), we have

$$\begin{aligned} A\left(\frac{a(p-1)}{r}, \frac{b(p-1)}{s}, m, l, \lambda\right) &\equiv \left(\frac{p}{p-1}\right)^w \sum_{\chi} \binom{\psi^a\chi}{\chi}^m \binom{\xi^b\chi}{\chi}^l \chi(-1)^l\chi(\lambda) \pmod p \\ &\equiv \left(\frac{p}{p-1}\right)^w \sum_{\chi} \binom{\psi^a\chi}{\chi}^m \binom{\xi^b\chi}{\chi}^l \chi((-1)^l\lambda) \pmod p. \end{aligned}$$

Thus the fact $\frac{1}{p-1} \equiv -1 \pmod p$ completes the proof of the theorem. □

Proof of Theorem 1.3 Putting $S = \phi$ in each expression of [4, Theorem 1.8], we obtain

$${}_2F_1\left(\begin{matrix}\chi_4, & \chi_4^3\\ & \epsilon\end{matrix} \mid \frac{4}{3}\right) = \begin{cases} 0, & \text{if } p \equiv 2 \pmod 3;\\ \phi(6)\chi_4(-1)\left[\binom{\phi}{\chi_3}+\binom{\phi}{\chi_3^2}\right], & \text{if } p \equiv 1 \pmod 3\end{cases}$$

$${}_2F_1\left(\begin{matrix}\chi_4, & \chi_4^3\\ & \epsilon\end{matrix} \mid -\frac{1}{3}\right) = \begin{cases} 0, & \text{if } p \equiv 2 \pmod 3;\\ \phi(6)\left[\binom{\phi}{\chi_3}+\binom{\phi}{\chi_3^2}\right], & \text{if } p \equiv 1 \pmod 3\end{cases}$$

$${}_2F_1\left(\begin{matrix}\chi_4, & \chi_4^3\\ & \epsilon\end{matrix} \mid 4\right) = \begin{cases} 0, & \text{if } p \equiv 2 \pmod 3;\\ \chi_4(12)\left[\binom{\phi}{\chi_3}+\binom{\phi}{\chi_3^2}\right], & \text{if } p \equiv 1 \pmod 3\end{cases}$$

$${}_2F_1\left(\begin{matrix}\chi_4, & \chi_4\\ & \epsilon\end{matrix} \mid \frac{1}{4}\right) = \begin{cases} 0, & \text{if } p \equiv 2 \pmod 3;\\ \chi_4(3)\left[\binom{\phi}{\chi_3}+\binom{\phi}{\chi_3^2}\right], & \text{if } p \equiv 1 \pmod 3),\end{cases}$$

respectively. Together with Theorem 1.2 and [8, Section 3.1], the first two expressions yield (i) and the last two complete the proof of (ii). □

In view of [6, Theorem 1.4], we can also deduce the following congruences similar to Theorem 1.3.

Corollary 3.1 *Let* $p = a_4^2 + b_4^2 \equiv 1 \pmod 4$*, where* $a_4 \equiv -\phi(2) \pmod 4$*. If* $d = \frac{p-1}{4}$*, then*

$$(i)\ A\left(d, d, 1, 1, -\frac{1}{9}\right) \equiv \chi_4(-1)A\left(d, d, 1, 1, -\frac{8}{9}\right) \equiv \phi(3)A(d, d^3, 1, 1, -9) \equiv -2a_4\phi(3)\chi_4(-1) \pmod p$$

$$(ii)\ A(d, d^3, 1, 1, \frac{1}{8}) \equiv \chi_4(-2)A\left(d, d^3, 1, 1, 8\right) \equiv \chi_4(-4)A\left(d, d^3, 1, 1, -\frac{9}{8}\right) \equiv -2a_4\chi_4(-2) \pmod p$$

Similarly, [7, Theorem 1.7 and Corollary 3.4] provides the following corollary.

Corollary 3.2 *If* $p \equiv 1 \pmod 3$ *and* $e = \frac{p-1}{3}$*, then*

$$(i)\ A(e, e, 1, 1, -\frac{9}{8}) \equiv A\left(e, e, 1, 1, \frac{1}{8}\right) \equiv A(e, e^2, 1, 1, -9) \equiv -2\mathrm{Re}\left[\chi_3(4)J(\phi, \chi_3^2)\right] \pmod p$$

$$(iii)\ A(e, e^2, 1, 1, -\frac{1}{9}) \equiv -2\,\mathrm{Re}\left[\chi_3(36)\mathrm{J}(\phi, \chi_3^2)\right] \pmod p$$

4 Preliminaries on *p*-adic gamma function

In this section we recall some notations and preliminary results required to prove Theorem 1.4. We begin with properties of p-adic Gamma function Γ_p (for details, see [19]). For $n \in \mathbb{N}$, the p-adic Gamma function is defined as

$$\Gamma_p(n) = (-1)^n \prod_{0<j<n, p\nmid j} j.$$

Extend Γ_p to all $x \in \mathbb{Z}_p$ by setting $\Gamma_p(0) = 1$ and

$$\Gamma_p(x) = \lim_{n \to x} \Gamma(n),$$

where n runs through any sequence of positive integers which p-adically approaches x. The following lemma provides some main properties of Gamma function, which are easy consequences of its definition.

Lemma 4.1 *[1,19] Let* $p \geq 5$ *be a prime. If* $x, y, z \in \mathbb{Z}_p$ *with* $|z| \leq |p|$*, where* $|\cdot|$ *is the* p*-adic norm. Then*

(a) $\Gamma_p(x+1) = \begin{cases} -x\Gamma_p(x), & \text{if } |x| = 1; \\ -\Gamma_p(x), & \text{if } |x| < 1. \end{cases}$

(b) $\Gamma_p(x)\Gamma_p(1-x) = (-1)^{Re(x)}$*, where* $Re(x)$ *denotes the representative of* $x \pmod p$ *in the set* $\{1, 2, \ldots, p\}$*.*

(c) if $x \equiv y \pmod{p^n}$*, then* $\Gamma_p(x) \equiv \Gamma(y) \pmod{p^n}$*;* $n \in \mathbb{N}$*.*

(d) $n! = (-1)^{n+1}\Gamma_p(n+1)$*;* $0 \leq n \leq (p-1)$*.*

(e) $\Gamma_p'(x+z) \equiv \Gamma_p'(x) \pmod p$*.*

(f) $\Gamma_p(x+z) \equiv \Gamma_p(x) + z\Gamma_p'(x) \pmod{p^2}$*.*

We consider the logarithmic derivative given by $G(x) := \frac{\Gamma_p'(x)}{\Gamma_p(x)}$. It is easy to see that if $x \in \mathbb{Z}_p$, then $G(x) \in \mathbb{Z}_p$. Moreover if $x \in \mathbb{Z}_p$ and $|x| = 1$, then the logarithmic derivative $G(x)$ satisfy the following nice relation

$$G(x+1) - G(x) = \frac{1}{x}. \tag{4.1}$$

Finally, we recall some background on Gauss sums. Let $\pi \in \mathbb{C}_p$ be a fixed root of $x^{p-1}+p=0$, and we let ζ_p be the unique pth root of unity in $\mathbb{C}_p$ such that $\zeta_p \equiv 1+\pi \pmod{\pi^2}$. For a multiplicative character χ, we define the Gauss sum to be

$$g(\chi)=\sum_{x=0}^{p-1}\chi(x)\zeta_p^x.$$

The following lemma provide some well-known properties of Gauss sums:

Lemma 4.2 *[8] For characters χ, χ_1, χ_2 on $\mathbb{F}_p$, we have*

(a) $g(\chi)g(\overline{\chi})=\chi(-1)p$.

(b) $J(\chi_1,\chi_2)=\begin{cases}\frac{g(\chi_1)g(\chi_2)}{g(\chi_1\chi_2)}, & \text{if } \chi_1\chi_2\neq\varepsilon;\\ -\chi_1(-1), & \text{if } \chi_1\chi_2=\varepsilon \quad \text{and} \quad \chi_1\neq\varepsilon, \chi_2\neq\varepsilon.\end{cases}$

Let ω denote the Teichmüller character, then ω can be defined uniquely by the property that $\omega(x)\equiv x \pmod{p}$ for $x=0,\ldots,p-1$. In this context, the Gross-Koblitz formula [15] states that

$$g(\overline{\omega}^j)=-\pi^j\Gamma_p\left(\frac{j}{p-1}\right), \quad 0\leq j\leq p-2. \tag{4.2}$$

5 Proof of Theorem 1.4

The main aim of this section is to prove Theorem 1.4. Following the approach of [1], we break the proof into a number of lemmas, and then combine them to prove the result.

Lemma 5.1 *Let $p>3$ be a prime such that $p\equiv 1 \pmod{r}$ and $ord_p(\lambda)>0$. If $T\in\widehat{\mathbb{F}_q^\times}$ is a generator of the character group, then*

$$-p^{w-1}{}_wF_{w-1}\left(\begin{matrix}T^{\frac{p-1}{r}}, & T^{\frac{p-1}{r}}, & \ldots, & T^{\frac{p-1}{r}}\\ & \epsilon, & \ldots, & \epsilon\end{matrix}\mid(-1)^l\lambda\right)$$
$$\equiv\frac{1}{\Gamma_p\left(\frac{1}{r}\right)^w}\sum_{j=0}^{\frac{p-1}{r}}\frac{\lambda^{pj}\Gamma_p\left(\frac{1}{r}+j\right)^w}{(-1)^{jl}\Gamma_p(1+j)^w}\left\{1+p\left\{1+wj\left\{G\left(\frac{1}{r}+j\right)-G(1+j)\right\}\right\}\right\} \pmod{p^2}.$$

Proof By definition of Gaussian hypergeometric series, we have

$${}_wF_{w-1}\left(\begin{matrix}T^{\frac{p-1}{r}}, & T^{\frac{p-1}{r}}, & \ldots, & T^{\frac{p-1}{r}}\\ & \epsilon, & \ldots, & \epsilon\end{matrix}\mid(-1)^l\lambda\right)$$
$$=\frac{p}{p-1}\sum_{\chi}\binom{T^{\frac{p-1}{r}}\chi}{\chi}^w\chi(-1)^l\chi(\lambda)$$
$$=\frac{1}{p^{w-1}(p-1)}\sum_{\chi}J\left(T^{\frac{p-1}{r}}\chi,\bar{\chi}\right)^w\chi(-1)^{l+w}\chi(\lambda).$$

By Lemma 4.2, we obtain

$$p^{w-1}{}_wF_{w-1}\left(\begin{matrix}T^{\frac{p-1}{r}}, & T^{\frac{p-1}{r}}, & \ldots, & T^{\frac{p-1}{r}}\\ & \epsilon, & \ldots, & \epsilon\end{matrix}\mid(-1)^l\lambda\right)$$
$$=\frac{1}{p-1}\sum_{j=0}^{p-2}J(\bar{\omega}^{\frac{p-1}{r}}\bar{\omega}^{-j},\bar{\omega}^j)^w\bar{\omega}^j(-1)^{l+w}\omega^j(\lambda)$$

$$= \frac{1}{p-1}\sum_{j=0}^{p-2}\frac{g(\bar{\omega}^{\frac{p-1}{r}-j})^w g(\bar{\omega}^j)^w}{g(\bar{\omega}^{\frac{p-1}{r}})^w}\bar{\omega}^{j(l+w)}(-1)\omega^j(\lambda)$$

$$= \frac{1}{p-1}\sum_{j=0}^{\frac{p-1}{r}}\frac{g(\bar{\omega}^{\frac{p-1}{r}-j})^w g(\bar{\omega}^j)^w}{g(\bar{\omega}^{\frac{p-1}{r}})^w}\bar{\omega}^{j(l+w)}(-1)\omega^j(\lambda)$$
$$+ \frac{1}{p-1}\sum_{j=\frac{p-1}{r}+1}^{p-2}\frac{g(\bar{\omega}^{\frac{r+1}{r}(p-1)-j})^w g(\bar{\omega}^j)^w}{g(\bar{\omega}^{\frac{p-1}{r}})^w}\bar{\omega}^{j(l+w)}(-1)\omega^j(\lambda),$$

where ω is the Teichmüller character. Using the Gross-Koblitz formula (4.2) and the fact that $\pi^{w(p-1)} = (-p)^w$, we deduce

$$p^{w-1}{}_wF_{w-1}\left(\begin{matrix} T^{\frac{p-1}{r}}, & T^{\frac{p-1}{r}}, & \ldots, & T^{\frac{p-1}{r}} \\ & \epsilon, & \ldots, & \epsilon \end{matrix} \mid (-1)^l\lambda\right)$$
$$= \frac{(-1)^w}{p-1}\sum_{j=0}^{\frac{p-1}{r}}\frac{\Gamma_p(\frac{1}{r}-\frac{j}{p-1})^w\Gamma_p(\frac{j}{p-1})^w}{\Gamma_p(\frac{1}{r})^w}\bar{\omega}^{j(l+w)}(-1)\omega^j(\lambda)$$
$$+ \frac{(-1)^w}{p-1}\sum_{j=\frac{p-1}{r}+1}^{p-2}\frac{\pi^{w(p-1)}\Gamma_p(\frac{r+1}{r}-\frac{j}{p-1})^w\Gamma_p(\frac{j}{p-1})^w}{\Gamma_p(\frac{1}{r})^w}\bar{\omega}^{j(l+w)}(-1)\omega^j(\lambda)$$
$$\equiv \frac{1}{p-1}\sum_{j=0}^{\frac{p-1}{r}}\frac{(-1)^w\Gamma_p(\frac{1}{r}+\frac{j}{1-p})^w\Gamma_p(\frac{-j}{1-p})^w}{\Gamma_p(\frac{1}{r})^w}\bar{\omega}^{j(l+w)}(-1)\omega^j(\lambda) \pmod{p^2}.$$

It is known that $\frac{j}{1-p} \equiv j + jp \pmod{p^2}$ and $\omega(\lambda) = \lambda^p \pmod{p^2}$. Using these together with Lemma 4.1 (b), we have

$$p^{w-1}{}_wF_{w-1}\left(\begin{matrix} T^{\frac{p-1}{r}}, & T^{\frac{p-1}{r}}, & \ldots, & T^{\frac{p-1}{r}} \\ & \epsilon, & \ldots, & \epsilon \end{matrix} \mid (-1)^l\lambda\right)$$
$$\equiv \frac{1}{p-1}\sum_{j=0}^{\frac{p-1}{r}}\frac{(-1)^w\Gamma_p(\frac{1}{r}+j+jp)^w\Gamma_p(-j-jp)^w}{\Gamma_p(\frac{1}{r})^w}\bar{\omega}^{j(l+w)}(-1)\omega^j(\lambda) \pmod{p^2}$$
$$\equiv \frac{1}{p-1}\sum_{j=0}^{\frac{p-1}{r}}\frac{\Gamma_p(\frac{1}{r}+j+jp)^w(-1)^{jl}}{\Gamma_p(\frac{1}{r})^w\Gamma_p(1+j+jp)^w}\lambda^{jp} \pmod{p^2}$$
$$\equiv -\frac{(1+p)}{\Gamma_p(\frac{1}{r})^w}\sum_{j=0}^{\frac{p-1}{r}}\frac{\Gamma_p(\frac{1}{r}+j+jp)^w}{\Gamma_p(1+j+jp)^w}(-1)^{jl}\lambda^{jp} \pmod{p^2}.$$

Finally, Lemma 4.1 (f) yields

$$-p^{w-1}{}_wF_{w-1}\left(\begin{matrix} T^{\frac{p-1}{r}}, & T^{\frac{p-1}{r}}, & \ldots, & T^{\frac{p-1}{r}} \\ & \epsilon, & \ldots, & \epsilon \end{matrix} \mid (-1)^l\lambda\right)$$
$$\equiv \frac{(1+p)}{\Gamma_p(\frac{1}{r})^w}\sum_{j=0}^{\frac{p-1}{r}}\frac{\Gamma_p(\frac{1}{r}+j)^w\left\{1+wjpG(\frac{1}{r}+j)\right\}}{\Gamma_p(1+j)^w\left\{1+wjpG(1+j)\right\}}(-1)^{jl}\lambda^{jp} \pmod{p^2}.$$

Multiplying numerator and denominator by $\{1 - wjpG(1+j)\}$, and then simplifying we complete the proof of the lemma. □

Lemma 5.2 *Let $p > 3$ be a prime for which $ord_p(\lambda) > 0$, then*

$$A\left(\frac{(r-1)(p-1)}{r}, \frac{p-1}{r}, m, m(r-1), \lambda^p\right) \equiv \frac{(-1)^{mr}}{\Gamma_p(\frac{1}{r})^{mr}} \sum_{j=0}^{\frac{p-1}{r}} \frac{\Gamma_p(\frac{1}{r}+j)^{mr}}{\Gamma_p(1+j)^{mr}} (-1)^{jm(r-1)} \lambda^{pj} \pmod{p^2}.$$

Proof Using Lemma 4.1 (d) and (b), we have

$$\begin{aligned} &A\left(\frac{(r-1)(p-1)}{r}, \frac{p-1}{r}, m, m(r-1), \lambda^p\right) \\ &= \sum_{j=0}^{\frac{p-1}{r}} \binom{\frac{(r-1)(p-1)}{r}+j}{j}^m \binom{\frac{p-1}{r}}{j}^{m(r-1)} \lambda^{pj} \\ &= \sum_{j=0}^{\frac{p-1}{r}} \frac{(\frac{(r-1)(p-1)}{r}+j)!^m}{j!^m \frac{(r-1)(p-1)}{r}!^m} \times \frac{(\frac{p-1}{r})!^{m(r-1)}}{j!^{m(r-1)} (\frac{p-1}{r}-j)!^{m(r-1)}} \lambda^{pj} \\ &= \frac{\Gamma_p(\frac{p-1}{r}+1)^{m(r-1)}}{(-1)^{mr}\Gamma_p(\frac{(r-1)(p-1)}{r}+1)^m} \sum_{j=0}^{\frac{p-1}{r}} \frac{\Gamma_p(\frac{(r-1)(p-1)}{r}+j+1)^m}{\Gamma_p(1+j)^{mr}\Gamma_p(\frac{p-1}{r}-j+1)^{m(r-1)}} \lambda^{pj} \\ &= \frac{(-1)^{mr}}{\Gamma_p(\frac{r-1}{r}p+\frac{1}{r})^m \Gamma_p(\frac{1}{r}-\frac{p}{r})^{m(r-1)}} \sum_{j=0}^{\frac{p-1}{r}} \frac{\Gamma_p(\frac{r-1}{r}p+j+\frac{1}{r})^m \Gamma_p(\frac{1}{r}+j-\frac{p}{r})^{m(r-1)}}{\Gamma_p(1+j)^{mr}(-1)^{mj(r-1)}} \lambda^{pj}. \end{aligned}$$

Finally, using Lemma 4.1 (f) and then simplifying we complete the proof of the lemma. □

Proof of Theorem 1.4 If $l = m(r-1)$, then Lemma 5.1 yields

$$\begin{aligned} &p^{mr-1}{}_{mr}F_{mr-1}\left(\begin{matrix} T^{\frac{p-1}{r}}, & T^{\frac{p-1}{r}}, & \cdots, & T^{\frac{p-1}{r}} \\ & \epsilon, & \cdots, & \epsilon \end{matrix} \mid (-1)^{m(r-1)}\lambda\right) \\ &\equiv -\frac{1}{\Gamma_p(\frac{1}{r})^{mr}} \sum_{j=0}^{\frac{p-1}{r}} \frac{\Gamma_p(\frac{1}{r}+j)^{mr}}{\Gamma_p(1+j)^{mr}} (-1)^{jm(r-1)}\lambda^{pj} - \frac{p}{\Gamma_p(\frac{1}{r})^{mr}} \sum_{j=0}^{\frac{p-1}{r}} \frac{\Gamma_p(\frac{1}{r}+j)^{mr}}{\Gamma_p(1+j)^{mr}} \\ &\times \left\{1 + wj\left\{G(\frac{1}{r}+j) - G(1+j)\right\}\right\} (-1)^{jm(r-1)}\lambda^{pj} \pmod{p^2}. \end{aligned}$$

It is easy to deduce that

$$\binom{\frac{r-1}{r}(p-1)+j}{j}^m \binom{\frac{p-1}{r}}{j}^{m(r-1)} \equiv \frac{(-1)^{mr}}{\Gamma_p(\frac{1}{r})^{mr}} \frac{\Gamma_p(\frac{1}{r}+j)^{mr}}{\Gamma_p(1+j)^{mr}} (-1)^{jm(r-1)} \pmod{p}.$$

Again, repeated application of (4.1) yields

$$G\left(\frac{1}{r}+j\right) - G(1+j) \equiv G\left(\frac{p+1}{r}+j\right) - G(1+j) \equiv H_{\frac{p-r+1}{r}+j} - H_j \pmod{p}.$$

Thus Lemma 5.2 together with the fact $\lambda^p \equiv \lambda \pmod{p}$, we have

$$
\begin{aligned}
&(-p)^{mr-1}{}_{mr}F_{mr-1}\left(\begin{matrix}T^{\frac{p-1}{r}},\ T^{\frac{p-1}{r}},\ \ldots,\ T^{\frac{p-1}{r}} \\ \epsilon,\quad \ldots,\quad \epsilon\end{matrix}\ \middle|\ (-1)^{m(r-1)}\lambda\right)\\
&\quad\equiv A\left(\frac{r-1}{r}(p-1), \frac{p-1}{r}, m, m(r-1), \lambda^p\right)\\
&\quad\quad + pB\left(\frac{r-1}{r}(p-1), \frac{p-1}{r}, m, m(r-1), \lambda\right) \pmod{p^2}
\end{aligned}
$$

completing the proof of the theorem. □

6 More supercongruences

In [2], Ahlgren proved (1.4) and recorded certain new supercongruences. We also find some similar supercongruences. In the following theorem, we denote by $D_2(\lambda)$ the discriminant of the polynomial $f(x) = x^4 - 4x^2 - 4\lambda;\ \lambda \neq 0, -1$.

Theorem 6.1 *Let $p > 3$ be a prime and $t \in \{1, -1\}$.*

(i) If $p = a_4^2 + b_4^2 \equiv 1 \pmod 8$ and $a_4 \equiv -\phi(2) \equiv= -1 \pmod 4$ such that $p \nmid D_2(-1/9)$, then

$$
\begin{aligned}
&A\left(\frac{p-1}{2}, \frac{p-1}{2}, 1, 1, -\left(\frac{4\sqrt{2}}{2\sqrt{2}+3t}\right)^p\right) + pB\left(\frac{p-1}{2}, \frac{p-1}{2}, 1, 1, -\frac{4\sqrt{2}}{2\sqrt{2}+3t}\right)\\
&\quad\equiv -2a_4\phi(3+2\sqrt{2}t) \pmod{p^2}.
\end{aligned}
$$

(ii) If $p \nmid D_2(1/3)$, then

$$
\begin{aligned}
&A\left(\frac{p-1}{2}, \frac{p-1}{2}, 1, 1, -\left(\frac{4}{2+\sqrt{3}t}\right)^p\right) + pB\left(\frac{p-1}{2}, \frac{p-1}{2}, 1, 1, -\frac{4}{2+\sqrt{3}t}\right)\\
&\quad\equiv \begin{cases} 0 \pmod{\mathrm{p}^2}, & \text{if } p \equiv 11 \pmod{12};\\ -2a_3\phi(3+2\sqrt{3}t) \pmod{\mathrm{p}^2}, & \text{if } p = a_3^2 + 3b_3^2 \equiv 1 \pmod{12}\\ & \text{and } a_3 \equiv -1 \pmod 3).\end{cases}
\end{aligned}
$$

Proof In [18, Theorem 1.1], ${}_2F_1(\lambda)$ was evaluated for $\lambda \in \left\{\frac{4\sqrt{2}}{2\sqrt{2}+3}, \frac{4\sqrt{2}}{2\sqrt{2}-3}, \frac{4}{2+\sqrt{3}}, \frac{4}{2-\sqrt{3}}\right\}$. Combining these evaluations with Theorem 1.4, we obtain the supercongruences of Theorem 6.1. □

Acknowledgements

We thank Ken Ono for going through the initial draft of the paper and many helpful suggestions during the preparation of the paper. We are grateful to the referee for his/her helpful comments.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors have made equal contributions to the conception of the article. Both authors carried out the study of the generalized Apéry numbers and deduced statements and proofs of the results. GK has drafted the manuscript. Both authors read and approved the final manuscript.

Received: 12 September 2016 Accepted: 29 November 2016
Published online: 01 March 2017

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