Locus *Res. Number Theory(2017)3:7*
DOI 10.1007/s40993-016-0070-6

Research in Number Theory

RESEARCH Open Access

Conjugacy growth series for finitary wreath products

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Abstract

We examine the conjugacy growth series of all wreath products of the finitary permutation groups Sym(X) and Alt(X) for an infinite set X. We determine their asymptotics, and we characterize the limiting behavior between the Alt(X) and Sym(X) wreath products. In particular, their ratios form a limit if and only if the dimension of the symmetric wreath product is twice the dimension of the alternating wreath product.

Keywords: Partitions, Group theory, Representation theory

Mathematics Subject Classification: 05A17, 11P81, 20B30, 20B35, 20C32

1 Introduction and statement of results

We begin by defining the infinite finitary symmetric and alternating groups and their corresponding wreath products, and then we state our results regarding growth series identities.

For an infinite set X, the *finitary symmetric group* $\mathrm{Sym}(X)$ is the group of permutations of X with finite support. The *permutational wreath product* of a group H with $\mathrm{Sym}(X)$ is the group $H \wr_X \mathrm{Sym}(X) := H^{(X)} \rtimes \mathrm{Sym}(X)$ defined as follows:

(i) The group $H^{(X)}$ is the group of functions from X to H with finite support.
(ii) The action of permutations $f \in \mathrm{Sym}(X)$ on functions $\psi \in H^{(X)}$ is defined by

$$\psi \mapsto f(\psi) := \psi \circ f^{-1}.$$

(iii) Multiplication in the semi-direct product is defined for $\varphi, \psi \in H^{(X)}$ and $f, g \in \mathrm{Sym}(X)$ by

$$(\varphi, f)(\psi, g) = (\varphi f(\psi), fg).$$

The *finitary alternating group* $\mathrm{Alt}(X)$ is the subgroup of $\mathrm{Sym}(X)$ of permutations with even signature, and the permutational wreath product $H \wr_X \mathrm{Alt}(X)$ is defined as above. In this paper, we only consider permutational wreath products with finite group H. We now define some general terminology. For any group G generated by a set S, the *word length* $\ell_{G,S}(g)$ of any element $g \in G$ is the smallest nonnegative integer n such that there exist $s_1, \ldots, s_n \in S \cup S^{-1}$ with $g = s_1 \cdots s_n$. The *conjugacy length* $\kappa_{G,S}(g)$ is the smallest word

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length appearing in the conjugacy class of g. If n is any natural number, we denote by $\gamma_{G,S}(n) \in \mathbb{N} \cup \{0\} \cup \{\infty\}$ the number of conjugacy classes in G with smallest word length n. If $\gamma_{G,S}(n)$ is finite for all n, then we may define the conjugacy growth series of a group G with generating set S to be the following q-series:

$$C_{G,S}(q) := \sum_{[g]\in \mathrm{Conj}(G)} q^{\kappa_{G,S}(g)} = \sum_{n=0}^{\infty} \gamma_{G,S}(n) q^n,$$

where the first sum is over representatives of conjugacy classes of G. Bacher and de la Harpe [1] prove conjugacy growth series identities for sufficiently large[1] generating sets S of $\mathrm{Sym}(X)$, S' of $\mathrm{Alt}(X)$, and $S^{(W_S)}$ of $W_S = H_S \wr_X \mathrm{Sym}(X)$ relating the finitary permutation groups and their wreath products to the partition function. Explicitly, we have the fascinating identities

$$C_{\mathrm{Sym}(X),S}(q) = \sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \tag{1.1}$$

for the finitary symmetric group,

$$C_{\mathrm{Alt}(X),S'}(q) = \left(\sum_{n=0}^{\infty} p(n) q^n\right)\left(\sum_{m=0}^{\infty} p_e(m) q^m\right) = \frac{1}{2}\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2} + \frac{1}{2}\prod_{n=1}^{\infty} \frac{1}{1-q^{2n}} \tag{1.2}$$

for the finitary alternating group,[2] and

$$C_{W_S,S^{(W_S)}}(q) = \sum_{n=0}^{\infty} \gamma_{W_S,S^{(W_S)}}(n) q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{M_S}} \tag{1.3}$$

for wreath products $W_S = H_S \wr_X \mathrm{Sym}(X)$, where M_S is the number of conjugacy classes of H_S. From now on, we denote $\gamma_{W_S}(n) := \gamma_{W_S,S^{(W_S)}}(n)$ for convenience.

Remark Recall Dedekind's eta function $\eta(\tau) = q^{1/24} \prod_{n\geq 1}(1-q^n)$ for $\tau \in \mathcal{H}$, where $\mathcal{H}$ denotes the upper half complex plane and $q := e^{2\pi i \tau}$. Equation (1.2) can be written as the linear combination of eta-quotients

$$C_{\mathrm{Alt}(X),S'}(q) = \frac{1}{2} \cdot \frac{q^{1/12}}{\eta(\tau)^2} + \frac{1}{2} \cdot \frac{q^{1/12}}{\eta(2\tau)},$$

which is essentially the sum of a modular form of weight -1 and a modular form of weight $-\frac{1}{2}$, up to multiplication by $q^{1/12}$. Studying such linear combinations may shed light on properties of sums of mixed weight modular forms.

It is natural to consider the number $\gamma_{W_S}(n)$ as a function of the number of conjugacy classes M_S in order to study properties of the coefficients of the above q-series. Here we

[1] The condition that the generating sets are sufficiently large refers to the properties defined in Sect. 2.
[2] Recall that $p_e(m)$ denotes the number of partitions of m into an even number of parts.

obtain a universal recurrence for these numbers. This result requires the ordinary divisor function $\sigma_k(n) = \sum_{d|n} d^k$. We also must define, for $n \geq 2$, the polynomial

$$\widehat{F}_n(x_1, \ldots, x_{n-1}) := \sum_{\substack{m_1,\ldots,m_{n-1}\geq 0 \\ m_1+\cdots+(n-1)m_{n-1}=n}} (-1)^{m_1+\cdots+m_{n-1}} \cdot \frac{(m_1+\cdots+m_{n-1}-1)!}{m_1!\cdots m_{n-1}!} \cdot x_1^{m_1}\cdots x_{n-1}^{m_{n-1}}.$$

Remark The polynomials $\widehat{F}_n$ are fairly straightforward to compute using only the partitions of n; the first few are listed below.

$$\begin{aligned}
\widehat{F}_2(x_1) &= \frac{1}{2}x_1^2, \\
\widehat{F}_3(x_1, x_2) &= -\frac{1}{3}x_1^3 + x_1x_2, \\
\widehat{F}_4(x_1, x_2, x_3) &= \frac{1}{4}x_1^4 - x_1^2x_2 + \frac{1}{2}x_2^2 + x_1x_3, \\
\widehat{F}_5(x_1, x_2, x_3, x_4) &= -\frac{1}{5}x_1^5 + x_1^3x_2 - x_1^2x_3 - x_1x_2^2 + x_1x_4 + x_2x_3, \\
\widehat{F}_6(x_1, x_2, x_3, x_4, x_5) &= \frac{1}{6}x_1^6 - x_1^4x_2 + x_1^3x_3 + \frac{3}{2}x_1^2x_2^2 - x_1^2x_4 - 2x_1x_2x_3 + x_1x_5 - \frac{1}{3}x_2^3 \\
&\quad + x_2x_4 + \frac{1}{2}x_3^2.
\end{aligned}$$

Remark These polynomials have been used in earlier work [2,4] on divisors of modular forms and the Rogers–Ramanujan identities.

Theorem 1 *Let $\widehat{F}_n(x_1, \ldots, x_{n-1})$ be defined as above. Let H_S be a finite group with M_S conjugacy classes, X an infinite set, and $W_S = H_S \wr_X Sym(X)$ a wreath product generated by a sufficiently large set $S^{(W_S)}$. Then we have*

$$C_{W_S,S^{(W_S)}}(q) = \sum_{n=0}^{\infty} \gamma_{W_S}(n)q^n = \prod_{n=1}^{\infty} (1-q^n)^{-M_S},$$

where $\gamma_{W_S}(n)$ satisfies the recurrence relation

$$\gamma_{W_S}(n) = \widehat{F}_n\big(\gamma_{W_S}(1), \ldots, \gamma_{W_S}(n-1)\big) + \frac{M_S}{n} \cdot \sigma_1(n).$$

In recent work, Nekrasov and Okounkov obtained a different formula for the infinite products in Theorem 1 in terms of hook lengths of partitions. Let $\lambda \vdash L$ denote that λ is a partition of the number L. The hook length of a partition $\lambda = (\lambda_1, \ldots, \lambda_n) \vdash L$ is defined using the Ferrers diagram of λ. For example, Fig. 1 below is a Ferrers diagram of the partition $\lambda = (6, 4, 3, 1, 1) \vdash 15$, Fig. 2 represents a hook length of 4, and Fig. 3 shows all hook lengths associated to λ.

More generally, for each box v in the Ferrers diagram of a partition λ, its *hook length* $h_v(\lambda)$ is defined as the number of boxes u such that

(i) $u = v$,
(ii) u is in the same column as v and below v, or
(iii) u is in the same row as v and to the right of v.

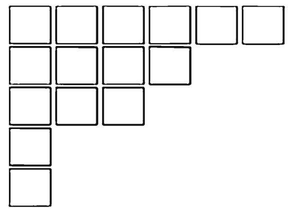

Fig. 1 Partition

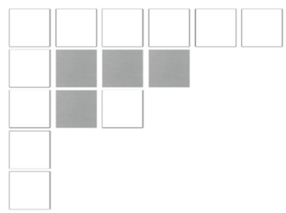

Fig. 2 Hook length

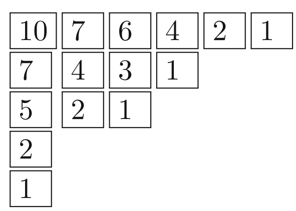

Fig. 3 Hook lengths

The *hook length multi-set* $\mathcal{H}(\lambda)$ is the set of all hook lengths of λ. Theorem 1 implies the following formula for $\gamma_{W_S}(n)$ in terms of hook lengths.

Corollary 2 *We have that*

$$\begin{aligned}\gamma_{W_S}(n) &= \widehat{F}_n\big(\gamma_{W_S}(1), \ldots, \gamma_{W_S}(n-1)\big) + \frac{M_S}{n} \cdot \sigma_1(n)\\ &= \sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)} \left(1 + \frac{M_S - 1}{h^2}\right).\end{aligned}$$

Remark Kostant observed [6] that the coefficients of the Nekrasov-Okounkov hook length identity are polynomials in the variable $z = 1 - M_S$, but he did not give an explicit formula for computing them.

Following Bacher's and de la Harpe's proofs of Eqs. (1.1), (1.2), and (1.3), we prove[3] the corresponding growth series identity for the wreath product $W_A = H_A \wr_X \mathrm{Alt}(X)$ with sufficiently large generating set $S^{(W_A)}$, namely

[3] See also Ian Wagner's arXiv preprint [9] on properties of Alt(X).

$$C_{W_A,S^{(W_A)}}(q) = \sum_{n=0}^{\infty} \gamma_{W_A,S^{(W_A)}}(n) q^n = \left(\frac{1}{2} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2} + \frac{1}{2} \prod_{n=1}^{\infty} \frac{1}{1-q^{2n}} \right)^{M_A}, \quad (1.4)$$

where M_A is the number of conjugacy classes of H_A. We provide the proof of Eq. (1.4) in Sect. 2. From now on, we denote $\gamma_{W_A}(n) := \gamma_{W_A,S^{(W_A)}}(n)$ for convenience.

In analogy with Theorem 1, one may ask if the coefficients $\gamma_{W_A}(n)$ in the alternating case can be seen as a function of the number of conjugacy classes M_A. We obtain a similar recurrence relation in this case.

Theorem 3 *Let $\widehat{F}_n(x_1, \ldots, x_{n-1})$ be defined as above. Let H_A be a finite group with M_A conjugacy classes, X an infinite set, and $W_A = H_A \wr_X Alt(X)$ a wreath product generated by a sufficiently large set $S^{(W_A)}$. Then we have*

$$C_{W_A,S^{(W_A)}}(q) = \sum_{n=0}^{\infty} \gamma_{W_A}(n) q^n = \left(\frac{1}{2} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2} + \frac{1}{2} \prod_{n=1}^{\infty} \frac{1}{1-q^{2n}} \right)^{M_A},$$

where $\gamma_{W_A}(n)$ satisfies the recurrence relation

$$\gamma_{W_A}(n) = \frac{1}{2^{M_A}} \sum_{k=0}^{M_A} \binom{M_A}{k} \Bigg(\widehat{F}_n\big(a_k(1), \ldots, a_k(n-1)\big) - \sum_{\delta \mid n} \delta \cdot \left[(-1)^{\delta}(k - M_A) - (k + M_A) \right] \Bigg),$$

and the a_k are defined by their generating function

$$\sum_{n=0}^{\infty} a_k(n) q^n := \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{2k} \left(1-q^{2n}\right)^{M_A - k}}.$$

Remark It may be possible to interpret the coefficients $\gamma_{W_A}(n)$ in terms of hook lengths from formulas of Han [5] or others, as in the symmetric case. The author does not make this connection here.

It is also natural to study the *modified exponential rate of conjugacy growth*[4] of a group G generated by a set S, namely

$$\widetilde{H}^{\text{conj}}_{G,S} = \limsup_{n \to \infty} \frac{\log \gamma_{G,S}(n)}{\sqrt{n}}.$$

It is useful to notice that $\exp\left(\widetilde{H}^{\text{conj}}_{G,S}\right)$ is the radius of convergence of the conjugacy growth series $C_{G,S}(q)$. For permutational wreath products, we apply a theorem of Cotron, Dicks and Fleming [3] on the asymptotic behavior of the generalized partition function defined in Sect. 2 [see Eqs. (2.1) and (2.2)]. Let $W_S = H_S \wr_X \text{Sym}(X)$ be a wreath product where H_S is a finite group, M_S is the number of conjugacy classes of H_S, and X is an infinite set. It is easy to see from Eq. (1.3) that the conjugacy growth series of such a wreath product is the generating function of the generalized partition function $p(n)_{\mathbf{e}}$ for the vector $\mathbf{e} = (M_S)$. This implies the following corollary.

[4]Cotron, Dicks, and Fleming [3] modify Bacher's and de la Harpe's definition [1] by changing the denominator from n to $\sqrt{n}$. With denominator n, most of the growth series that we study have exponential rate of conjugacy growth zero.

Corollary 4 *Let $W_S = H_S \wr_X Sym(X)$ be a wreath product where H_S is a finite group, M_S is the number of conjugacy classes of H_S, and X is an infinite set. If $S^{(W_S)}$ is a sufficiently large generating set of W_S, then we have*

$$\gamma_{W_S}(n) \sim \left(\frac{M_S^{\frac{1+M_S}{4}}}{2^{\frac{5+3M_S}{4}} 3^{\frac{1+M_S}{4}} n^{\frac{3+M_S}{4}}} \right) e^{\pi \sqrt{\frac{2nM_S}{3}}}.$$

We now give the modified exponential rate of conjugacy growth for wreath products in the symmetric case using this asymptotic formula.

Corollary 5 *The modified exponential rate of conjugacy growth for the group $W_S = H_S \wr_X Sym(X)$ defined above is*

$$\widetilde{H}^{conj}_{W_S} = \pi \sqrt{\frac{2nM_S}{3}} + \frac{1}{\sqrt{n}} \cdot \log \left(\frac{M_S^{\frac{1+M_S}{4}}}{2^{\frac{5+3M_S}{4}} 3^{\frac{1+M_S}{4}} n^{\frac{3+M_S}{4}}} \right).$$

We can also apply the theorem to wreath products in the alternating case using Eq. (1.4).

Corollary 6 *Let $W_A = H_A \wr_X Alt(X)$ be a wreath product where H_A is a finite group, M_A is the number of conjugacy classes of H_A, and X is an infinite set. If $S^{(W_A)}$ is a sufficiently large generating set of W_A, then we have*

$$\gamma_{W_A}(n) \sim \left(\frac{M_A^{\frac{1+2M_A}{4}}}{2^{1+2M_A} 3^{\frac{1+2M_A}{4}} n^{\frac{3+2M_A}{4}}} \right) e^{2\pi \sqrt{\frac{nM_A}{3}}}.$$

We also give the modified exponential rate of conjugacy growth in the alternating case using the above asymptotic formula.

Corollary 7 *The modified exponential rate of conjugacy growth for the group $W_A = H_A \wr_X Alt(X)$ defined above is*

$$\widetilde{H}^{conj}_{W_A} = 2\pi \sqrt{\frac{nM_A}{3}} + \frac{1}{\sqrt{n}} \cdot \log \left(\frac{M_A^{\frac{1+2M_A}{4}}}{2^{1+2M_A} 3^{\frac{1+2M_A}{4}} n^{\frac{3+2M_A}{4}}} \right).$$

We are interested in finding relationships between wreath products of Sym(X) and wreath products of Alt(X). Let $W_S = H_S \wr_X \mathrm{Sym}(X)$ and $W'_S = H'_S \wr_X \mathrm{Sym}(X)$ be two wreath products of Sym(X), where H_S, H'_S are finite groups with M_S, M'_S conjugacy classes respectively. Let $W_A = H_A \wr_X \mathrm{Alt}(X)$ and $W'_A = H'_A \wr_X \mathrm{Alt}(X)$ be two wreath products of Alt(X), where H_A, H'_A are finite groups with M_A, M'_A conjugacy classes respectively.

Question 1 *What is the asymptotic behavior of the following ratios?*

$$(1) \quad \frac{\gamma_{W_S}(n)}{\gamma_{W'_S}(n)}, \qquad (2) \quad \frac{\gamma_{W_S}(n)}{\gamma_{W_A}(n)}, \qquad (3) \quad \frac{\gamma_{W_A}(n)}{\gamma_{W_S}(n)}, \qquad (4) \quad \frac{\gamma_{W_A}(n)}{\gamma_{W'_A}(n)}.$$

In particular, when do the ratios approach some nonzero finite number?

The asymptotic behavior of the ratios follows from Corollaries 4 and 6.

Corollary 8 *Let W_S, W_S', W_A, and W_A' be groups as above. Then as $n \to \infty$, we have*

$$(1) \qquad \frac{\gamma_{W_S}(n)}{\gamma_{W_S'}(n)} \sim \left(\frac{M_S^{\frac{1+M_S}{4}}}{{M_S'}^{\frac{1+M_S'}{4}}} \right) \left[2^{\frac{3}{4}(M_S'-M_S)} (3n)^{\frac{M_S'-M_S}{4}} \right] e^{\pi \sqrt{\frac{2n}{3}} \left(\sqrt{M_S} - \sqrt{M_S'} \right)}.$$

$$(2) \qquad \frac{\gamma_{W_S}(n)}{\gamma_{W_A}(n)} \sim \left(\frac{M_S^{\frac{1+M_S}{4}}}{M_A^{\frac{1+2M_A}{4}}} \right) \left[2^{\frac{8M_A-3M_S-1}{4}} (3n)^{\frac{2M_A-M_S}{4}} \right] e^{\pi \sqrt{\frac{2n}{3}} \left(\sqrt{M_S} - \sqrt{2M_A} \right)}.$$

$$(3) \qquad \frac{\gamma_{W_A}(n)}{\gamma_{W_S}(n)} \sim \left(\frac{M_A^{\frac{1+2M_A}{4}}}{M_S^{\frac{1+M_S}{4}}} \right) \left[2^{\frac{1+3M_S-8M_A}{4}} (3n)^{\frac{M_S-2M_A}{4}=} \right] e^{\pi \sqrt{\frac{2n}{3}} \left(\sqrt{2M_A} - \sqrt{M_S} \right)}.$$

$$(4) \qquad \frac{\gamma_{W_A}(n)}{\gamma_{W_A'}(n)} \sim \left(\frac{M_A^{\frac{1+2M_A}{4}}}{{M_A'}^{\frac{1+2M_A'}{4}}} \right) \left[4^{(M_A'-M_A)} (3n)^{\frac{M_A'-M_A}{2}} \right] e^{2\pi \sqrt{\frac{n}{3}} \left(\sqrt{M_A} - \sqrt{M_A'} \right)}.$$

We now observe for which pairs (M_S, M_S'), (M_S, M_A), (M_A, M_S), and (M_A, M_A') these ratios asymptotically approach zero, infinity, or some nonzero finite number. Corollary 9 follows from the asymptotic behavior of the exponential functions in Corollary 8.

Corollary 9 *Let W_S, W_S', W_A, and W_A' be groups as above. Then as $n \to \infty$, we have the following asymptotic behavior.*

(1) If $M_S < M_S'$, then $\frac{\gamma_{W_S}(n)}{\gamma_{W_S'}(n)} \sim 0$. If $M_S > M_S'$, then $\frac{\gamma_{W_S}(n)}{\gamma_{W_S'}(n)} \sim \infty$.
If $M_S = M_S'$, then $\frac{\gamma_{W_S}(n)}{\gamma_{W_S'}(n)} \sim 1$.

(2) If $M_S < 2M_A$, then $\frac{\gamma_{W_S}(n)}{\gamma_{W_A}(n)} \sim 0$. If $M_S > 2M_A$, then $\frac{\gamma_{W_S}(n)}{\gamma_{W_A}(n)} \sim \infty$.
If $M_S = 2M_A$, then $\frac{\gamma_{W_S}(n)}{\gamma_{W_A}(n)} \sim 2^{M_A}$.

(3) If $2M_A < M_S$, then $\frac{\gamma_{W_A}(n)}{\gamma_{W_S}(n)} \sim 0$. If $2M_A > M_S$, then $\frac{\gamma_{W_A}(n)}{\gamma_{W_S}(n)} \sim \infty$.
If $2M_A = M_S$, then $\frac{\gamma_{W_A}(n)}{\gamma_{W_S}(n)} \sim \frac{1}{2^{M_A}}$.

(4) If $M_A < M_A'$, then $\frac{\gamma_{W_A}(n)}{\gamma_{W_A'}(n)} \sim 0$. If $M_A > M_A'$, then $\frac{\gamma_{W_A}(n)}{\gamma_{W_A'}(n)} \sim \infty$.
If $M_A = M_A'$, then $\frac{\gamma_{W_A}(n)}{\gamma_{W_A'}(n)} \sim 1$.

Moreover, the converses of all of the above statements hold as well.

Given any two wreath products of Sym(X) or Alt(X), Corollary 9 guarantees the asymptotic behavior of the ratios between the coefficients of their conjugacy growth series. In other words, for any two wreath products W and W', we know the expected relationship between the number of conjugacy classes of H in W and the number of conjugacy classes of H' in W' with minimal word length n for any n.

Remark Although we know the asymptotic behavior of the above ratios, this does not mean that the ratios of the coefficients are always exactly equal to the above values.

For example, consider the wreath products $W_S = H_S \wr_X \mathrm{Sym}(X)$ and $W_A = H_A \wr_X \mathrm{Alt(X)}$, where H_S, H_A are finite groups with $M_S = 10, M_A = 5$ conjugacy classes respectively. We expect the ratio of the coefficients of W_S to the coefficients of W_A to be asymptotic to $2^5 = 32$. We compute the following coefficients with Maple.

n	$\gamma_{W_S}(n)$	$\gamma_{W_A}(n)$	$\frac{\gamma_{W_S}(n)}{\gamma_{W_A}(n)}$
1	10	5	2
10	1605340	176963	9.071613840
100	$0.2333013623 \times 10^{28}$	$0.7541087996 \times 10^{26}$	30.93736108
200	$0.1067904403 \times 10^{42}$	$0.3346942881 \times 10^{40}$	31.90686071
300	$0.4721905614 \times 10^{52}$	$0.1476229954 \times 10^{51}$	31.98624714
400	$0.5248644122 \times 10^{61}$	$0.1640339890 \times 10^{60}$	31.99729613
500	$0.5369981415 \times 10^{69}$	$0.1678152777 \times 10^{68}$	31.99935959

2 Proofs

We give the proofs of Eq. (1.4) and Theorems 1 and 3 here. We also explain what it means for a generating set to be sufficiently large and give remarks on Corollaries 2 and 6.

A set S of transpositions of a set X is called *partition-complete* (*PC*) [1] if

(i) The transposition graph $\Gamma(S)$ is connected, and
(ii) For every partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash L$, $\Gamma(S)$ contains a forest of k trees with $\lambda_1 + 1, \ldots, \lambda_k + 1$ vertices respectively.

For the corresponding property of *partition-complete for wreath products* (*PCwr*) [1], we must first establish more notation. Let X be an infinite set, H a group, and $W = H \wr_X \mathrm{Sym}(X)$. The group W acts naturally on the set $H \times X$; namely, for $(\varphi, f) \in W$, the action is defined by

$$(h, x) \mapsto \big(\varphi(f(x))h, f(x)\big).$$

For $a \in H\backslash\{1\}$ and $u \in X$, we let $\varphi_u^a \in W$ denote the permutation that maps $(h, x) \in H \times X$ to (ah, u) if $x = u$, and to (h, x) otherwise. Then $(\varphi_u^a)_{a \in H\backslash\{1\}, u \in X}$ generates the group $H^{(X)}$. Now, let $H_u := \{\varphi_u^a \mid a \in H \setminus \{1\}\}$, and define the subsets

$$T_H := \bigcup_{u \in X} H_u \subseteq H^{(X)},$$
$$T_X := \{(x \;\; y) \in \mathrm{Sym}(X) : x, y \in X \text{ are distinct}\} \subseteq \mathrm{Sym}(X).$$

Let $S_H \subset T_H$ and $S_X \subset T_X$ be subsets, and let $S = S_H \sqcup S_X \subseteq W$. Such a set S is said to be *PCwr* if

(i) The transposition graph $\Gamma(S_X)$ is connected, and
(ii) For all $L \geq 0$ and partitions $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash L$, $\Gamma(S_X)$ contains a forest of k trees $T_1, \ldots, T_k$, with T_i having λ_i vertices, including one vertex $x^{(i)}$ such that $\varphi^a_{x^{(i)}} \in S_H$ for all $a \in H\backslash\{1\}$.

Remark The conditions *PC* and *PCwr* essentially require the generating set S to contain "enough" transpositions to represent all possible partitions in its transposition graph.

Proof of equation (1.4) This proof follows[5] from the proofs of eqs. (1.2) and (1.3) in [1]. For each $w = (\phi, \sigma) \in W_A = H_A \wr_X \mathrm{Alt}(X)$, we can split σ into a product of an even number of cycles of even length, denoted σ_e, and a product of cycles of odd length, denoted σ_o, so that $w = (\phi, \sigma_e\sigma_o)$. Let $(H_A)_*$ denote the set of conjugacy classes of H_A; we write $1 \in (H_A)_*$ for

[5]A careful description of the conjugacy classes of wreath products can also be found in Macdonald's 1995 book, Symmetric Functions and Hall Polynomials.

the class $\{1\} \in H_A$. To each conjugacy class in W_A we associate an $(H_A)_*$-indexed family of partitions. Using the same notation as in [1], we associate the conjugacy classes in H_A to the family of partitions

$$\left(\lambda^{(1)}, \nu^{(1)}; \left(\mu^{(\eta)}, \gamma^{(\eta)}\right)_{\eta\in(H_A)_*\backslash 1}\right),$$

where $\nu^{(1)}$ and $\gamma^{(\eta)}$ each have an even number of positive parts, in the following way.

Let $X^{(w)}$ be the finite subset of X that is the union of the supports of ϕ and σ. Let σ be the product of the disjoint cycles $c_1, ..., c_k$, where $c_i = \left(x_1^{(i)}, x_2^{(i)}, ..., x_{\nu_i}^{(i)}\right)$ with $x_j^{(i)} \in X^{(w)}$ and $\nu_i =$ length(c_i). We include cycles of length 1 for each $x \in X$ such that $x \in \sup(\phi)$ and $x \notin \sup(\sigma)$, so that

$$X^{(w)} = \bigsqcup_{1\le i\le k} \sup(c_i).$$

Define $\eta_*^w(c_i) \in (H_A)_*$ to be the conjugacy class of the product $\phi\left(x_{\nu_i}^{(i)}\right)\phi\left(x_{\nu i-1}^{(i)}\right)\cdots\phi\left(x_1^{(i)}\right) \in H_A$. For $\eta \in (H_A)_*$ and $\ell \ge 1$, let $m_\ell^{w,\eta}$ denote the number of cycles c in $\{c_1, ..., c_k\}$ such that length$(c) = \ell$ and $\eta_*^w(c) = \eta$. Let $\mu^{w,\eta} \vdash n^{w,\eta}$ be the partition with $m_\ell^{w,\eta}$ parts equal to ℓ, for all $\ell \ge 1$. Note that

$$\sum_{\eta\in(H_A)_*} n^{w,\eta} = \sum_{\eta\in(H_A)_*,\ \ell\ge1} \ell m_\ell^{w,\eta} = |X^{(w)}|.$$

Also observe that the partition $\mu^{w,1}$ does not contain parts of size 1, because if $\nu_i = 1$, then $\eta_*^w(c_i) \neq 1$. Using the same notation as above, let $\lambda^{w,1}$ be the partition with $m_\ell^{w,1}$ parts equal to $\ell - 1$. We can write $\sigma = \sigma_e\sigma_o$ as above, so $\mu^{w,1}$ and $\lambda^{w,1}$ each split into two partitions, one of which has an even number of parts. Define the *type* of w to be the family $\left(\lambda^{(1)}, \nu^{(1)}; \left(\mu^{(\eta)}, \gamma^{(\eta)}\right)_{\eta\in(H_A)_*\backslash 1}\right)$. Then two elements in W_A are conjugate if and only if they have the same type. Thus, each $(H_A)_*$-indexed family of partitions $\left(\lambda^{(1)}, \nu^{(1)}; \left(\mu^{(\eta)}, \gamma^{(\eta)}\right)_{\eta\in(H_A)_*\backslash 1}\right)$ is the type of one conjugacy class in W_A.

Consider an $(H_A)_*$-indexed family of partitions $\left(\lambda^{(1)}, \nu^{(1)}; \left(\mu^{(\eta)}, \gamma^{(\eta)}\right)_{\eta\in(H_A)_*\backslash 1}\right)$ and the corresponding conjugacy class in W_A. Let $u^{(1)}, v^{(1)}, u^{(\eta)}, v^{(\eta)}$ be the sums of the parts of $\lambda^{(1)}, \nu^{(1)}, \mu^{(\eta)}, \gamma^{(\eta)}$ respectively, and let $k^{(1)}, t^{(1)}, k^{(\eta)}, t^{(\eta)}$ be the number of parts of $\lambda^{(1)}, \nu^{(1)}, \mu^{(\eta)}, \gamma^{(\eta)}$ respectively.

Choose a representative $w = (\phi, \sigma)$ of this conjugacy class such that

$$\sigma = \prod_{i=1}^{k} c_i = \prod_{i=1}^{k}\left(x_1^{(i)}, x_2^{(i)}, ..., x_{\mu_i}^{(i)}\right)$$

and

$$\phi\left(x_j^{(i)}\right) = 1 \in H_A \quad \text{for all } j \in \{1, ..., \mu_i\} \quad \text{when } \eta_*^w(c_i) = 1,$$

$$\phi\left(x_j^{(i)}\right) = \begin{cases} 1 & \text{for all } j \in \{1, ..., \mu_i - 1\} \\ h \neq 1 & \text{for } j = \mu_i \end{cases} \quad \text{when } \eta_*^w(c_i) \neq 1.$$

Observe that

$$k = k^{(1)} + t^{(1)} + \sum_{\eta\in(H_A)_*\backslash 1}\left(k^{(\eta)} + t^{(\eta)}\right),$$

$$\left|X^{(w)}\right| = u^{(1)} + k^{(1)} + v^{(1)} + t^{(1)} + \sum_{\eta\in(H_A)_*\backslash 1}\left(u^{(\eta)} + v^{(\eta)}\right).$$

Hence, the contribution to $C_{W_A,S^{(W_A)}}(q)$ from $\left(\lambda^{(1)},\nu^{(1)};\left(\mu^{(\eta)},\gamma^{(\eta)}\right)_{\eta\in(H_A)_*\backslash 1}\right)$ is

$$\left(q^{u^{(1)}}q^{\nu^{(1)}}\prod_{\eta\in(H_A)_*\backslash 1}q^{u^{(\eta)}}q^{\nu^{(\eta)}}\right).$$

It follows that

$$\begin{aligned}
C_{W_A,S^{(W_A)}}(q) &= \left[\left(\prod_{u_1=1}^{\infty}\frac{1}{1-q^{u_1}}\right)\left(\frac{1}{2}\prod_{\nu_1=1}^{\infty}\frac{1}{1-q^{\nu_1}}+\frac{1}{2}\prod_{\nu_1=1}^{\infty}\frac{1}{1+q^{\nu_1}}\right)\right]\\
&\quad\times\prod_{\eta\in(H_A)_*\backslash 1}\left[\left(\prod_{u_\eta=1}^{\infty}\frac{1}{1-q^{u_\eta}}\right)\left(\frac{1}{2}\prod_{\nu_\eta=1}^{\infty}\frac{1}{1-q^{\nu_\eta}}+\frac{1}{2}\prod_{\nu_\eta=1}^{\infty}\frac{1}{1+q^{\nu_\eta}}\right)\right]\\
&= \left[\left(\frac{1}{2}\prod_{n_1=1}^{\infty}\frac{1}{1-q^{2n_1}}+\frac{1}{2}\prod_{n_1=1}^{\infty}\frac{1}{(1-q^{n_1})^2}\right)\right]\\
&\quad\times\prod_{\eta\in(H_A)_*\backslash 1}\left[\left(\frac{1}{2}\prod_{n_\eta=1}^{\infty}\frac{1}{1-q^{2n_\eta}}+\frac{1}{2}\prod_{n_\eta=1}^{\infty}\frac{1}{(1-q^{n_\eta})^2}\right)\right]\\
&= \left(\frac{1}{2}\prod_{k=1}^{\infty}\frac{1}{1-q^{2k}}+\frac{1}{2}\prod_{k=1}^{\infty}\frac{1}{(1-q^{k})^2}\right)^{|(H_A)_*|}.
\end{aligned}$$

The equality between the first and second line is given in the appendix of [1]. □

The *generalized partition function* $p(n)_{\mathbf{e}}$ is defined for the vector $\mathbf{e}=(e_1,\ldots,e_k)\in\mathbb{Z}^k$ by its generating function

$$\sum_{n=0}^{\infty}p(n)_{\mathbf{e}}q^n=\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{e_1}\cdots(1-q^{kn})^{e_k}}. \tag{2.1}$$

The following theorem gives an asymptotic formula for the generalized partition function, which was obtained by using properties of modular forms.[6]

Theorem (Cotron-Dicks-Fleming [3]) Let $\mathbf{e}=(e_1,\ldots,e_k)$ be any nonzero vector with nonnegative integer entries, and let $d:=\gcd\{m:e_m\neq 0\}$. Define the quantities

$$\gamma:=\gamma(\mathbf{e})=\sum_{m=1}^{k}e_{dm}\quad\text{and}\quad\delta:=\delta(\mathbf{e})=\sum_{m=1}^{k}\frac{e_{dm}}{m}.$$

Then as $n\to\infty$, we have that

$$p(dn)_{\mathbf{e}}\sim\frac{\lambda A^{\frac{1+\gamma}{4}}}{2\sqrt{\pi}\,n^{\frac{3+\gamma}{4}}}e^{2\sqrt{An}}, \tag{2.2}$$

where

$$\lambda:=\prod_{m=1}^{k}\left(\frac{m}{2\pi}\right)^{\frac{e_{dm}}{2}}\quad\text{and}\quad A:=\frac{\pi^2\delta}{6}.$$

[6]For background on modular forms, see [8].

Corollaries 4, 6, 8, and 9 all follow from the above theorem, since the conjugacy growth series for permutational wreath products correspond to the generalized partition function with vectors (M_S) in the symmetric case and $(2k, M_A - k)$, $0 \leq k \leq M_A$, in the alternating case.

A Remark on Corollary 6 By the binomial theorem applied to the conjugacy growth series in Eq. (1.4), we find that

$$\gamma_{W_A}(n) \sim \frac{1}{2^{M_A}} \sum_{k=0}^{M_A} \left[\frac{(4M_A - 3k)^{\frac{1+2M_A-k}{4}}}{2^{\frac{4M_A-3k+3}{2}} 3^{\frac{1+2M_A-k}{4}} n^{\frac{3+2M_A-k}{4}}} \cdot e^{2\pi\sqrt{\left(\frac{4M_A-3k}{12}\right)n}} \right].$$

But, intuitively, the summands corresponding to $k > 0$ grow much more slowly than the summand corresponding to $k = 0$, since the instance of k in the exponential function is negative. Therefore, the above sum is asymptotic to the $k = 0$ term, so we have

$$\gamma_{W_A}(n) \sim \frac{M_A^{\frac{1+2M_A}{4}}}{2^{1+2M_A} 3^{\frac{1+2M_A}{4}} n^{\frac{3+2M_A}{4}}} \cdot e^{2\pi\sqrt{\frac{nM_A}{3}}}.$$

□

We now introduce the proofs of Theorems 1 and 3. In a paper by Bruinier, Kohnen, and Ono [2], the universal polynomial F_n is defined as

$$\begin{aligned} F_n(x_1, \ldots, x_{n-1}) :=& -\frac{2x_1\sigma_1(n-1)}{n-1} \\ &+ \sum_{\substack{m_1,\ldots,m_{n-2}\geq 0 \\ m_1+\cdots+(n-2)m_{n-2}=n-1}} (-1)^{m_1+\cdots+m_{n-2}} \cdot \frac{(m_1+\cdots+m_{n-2}-1)!}{m_1!\cdots m_{n-2}!} \cdot x_2^{m_1}\cdots x_{n-1}^{m_{n-2}}, \end{aligned}$$

and it is used to define a recursion relation for coefficients of meromorphic modular forms on $SL_2(\mathbb{Z})$. Frechette and the author [4] modify this polynomial to the above $\widehat{F}_n$ and use it to define a recursion relation for coefficients of quotients of Rogers-Ramanujan-type q-series. Their proof surprisingly only requires properties of logarithmic derivatives applied to a q-series infinite product identity. The proof below is adapted from the proof in [4] and can be applied to any q-series infinite product identity, including the famous identity of Nekrasov and Okounkov [7].

Proof of Theorem 1 Define the q-series identity

$$F_r(q) := \sum_{n=0}^{\infty} p_n(r)q^n := \prod_{n=1}^{\infty}(1-q^n)^r$$

so that $p_n(r) = \gamma_{W_S}(n)$ and $r = -M_S$. We take logarithms of both sides to obtain

$$\begin{aligned} \log\left(1 + \sum_{n=1}^{\infty} p_n(r)q^n\right) &= \sum_{n=1}^{\infty} r\log(1-q^n) \\ &= -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{rq^{kn}}{k}, \end{aligned}$$

by the Taylor expansion for $\log(1-x)$. Then we take the derivatives of both sides to obtain

$$\frac{\sum_{n=1}^{\infty} np_n(r)q^{n-1}}{1+\sum_{n=1}^{\infty} p_n(r)q^n} = -\sum_{n=1}^{\infty} \sum_{d|n} rdq^{n-1} = -\sum_{n=1}^{\infty} r\sigma_1(n)q^{n-1},$$

so we have

$$\sum_{n=1}^{\infty} np_n(r)q^n = \left(-\sum_{n=1}^{\infty} r\sigma_1(n)q^n\right)\left(1+\sum_{n=1}^{\infty} p_n(r)q^n\right).$$

For convenience, define $b(n) := r\sigma_1(n)$. Expanding the right hand side and equating coefficients, we now have

$$0 = b(n) + b(n-1)p_1(r) + b(n-2)p_2(r) + \cdots + b(1)p_{n-1}(r) + np_n(r).$$

The symmetric power functions

$$s_i := X_1^i + \cdots + X_n^i$$

and the elementary symmetric functions[7]

$$\sigma_i = \sum_{1\le j_1\le\cdots\le j_i\le n} X_{j_1}\cdots X_{j_i}$$

exhibit a similar relationship; namely, we have the identity

$$0 = s_n - s_{n-1}\sigma_1 + s_{n-2}\sigma_2 - \cdots + (-1)^{n-1}s_1\sigma_{n-1} + (-1)^n\sigma_n. \tag{2.3}$$

Evaluating Eq. (2.3) at $(X_1,\ldots,X_n) = (\lambda(1,n),\ldots,\lambda(n,n))$, where $\lambda(j,n)$ are the roots of the polynomial

$$X^n + p_1(r)X^{n-1} + \cdots + p_{n-1}(r)X + p_n(r),$$

we have that $p_n(r) = (-1)^n\sigma_n$ for $n \ge 1$. Then we have $b(n) = s_n$. Using the fact that

$$s_n = n \sum_{\substack{m_1,\ldots,m_n\ge 0\\ m_1+\cdots+nm_n=n}} (-1)^{m_2+m_4+\cdots} \cdot \frac{(m_1+\cdots+m_n-1)!}{m_1!\cdots m_n!} \cdot \sigma_1^{m_1}\cdots\sigma_n^{m_n},$$

we see that $\frac{b(n)}{n}$ is exactly $\widehat{F}_n(p_1(r),\ldots,p_{n-1}(r))$ plus the additional term with $m_1 = \cdots = m_{n-1} = 0$ and $m_n = 1$. We arrive at the recursion

$$p_n(r) = \widehat{F}_n(p_1(r),\ldots,p_{n-1}(r)) - \frac{r}{n}\sigma_1(n).$$

Thus, we have

$$\gamma_{W_S}(n) = \widehat{F}_n(\gamma_{W_S}(1),\ldots,\gamma_{W_S}(n-1)) + \frac{M_S}{n}\sigma_1(n).$$

□

[7] Note that s_i and σ_i are both functions of $X_1, X_2, \ldots, X_n$.

Theorem 1 gives a recurrence formula for the coefficients $\gamma_{W_S}(n)$ of the conjugacy growth series of a permutational wreath product in which the group H_S has M_S conjugacy classes. Now, we consider the more general infinite product $\prod_{n\geq 1}(1-q^n)^r$ for any complex number r, and we ignore its implications for finite groups. Then the above proof also applies to the coefficients of the Nekrasov-Okounkov hook length formula [7]

$$\sum_{\lambda\in\mathcal{P}} x^{|\lambda|} \prod_{h\in\mathcal{H}(\lambda)} \left(1-\frac{z}{h^2}\right) = \prod_{k\geq 1}\left(1-x^k\right)^{z-1}$$

if we change variables $z \mapsto 1+r$ and $x \mapsto q := e^{2\pi i\tau}$ for $\tau \in \mathcal{H}$. The coefficients

$$\prod_{h\in\mathcal{H}(\lambda)} \left(1-\frac{z}{h^2}\right) = \prod_{h\in\mathcal{H}(\lambda)} \left(1-\frac{1+r}{h^2}\right)$$

of the infinite product

$$\prod_{k\geq 1}\left(1-x^k\right)^{z-1} = \prod_{n\geq 1}(1-q^n)^r$$

therefore satisfy the recurrence relation

$$\prod_{h\in\mathcal{H}(\lambda)} \left(1-\frac{1+r}{h^2}\right) = \gamma_{W_S}(n) = \widehat{F}_n\big(\gamma_{W_S}(1),\ldots,\gamma_{W_S}(n-1)\big) - \frac{r}{n}\sigma_1(n).$$

Although for $r \in \mathbb{C}\setminus\mathbb{Z}^+$ we can no longer observe the relationship between the number of conjugacy classes of H_S and the coefficients of the conjugacy growth series of $H_S \wr_X \mathrm{Sym}(X)$, we do obtain a simple recursion for the Nekrasov-Okounkov hook length formula which is independent of complex analysis and hook lengths.

Proof of Theorem 3 This proof closely follows the proof of Theorem 1. Define the q-series identity

$$F_{M_A}(q) := \sum_{n=0}^{\infty} P_n(M_A)q^n := \left(\frac{1}{2}\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^2} + \frac{1}{2}\prod_{n=1}^{\infty}\frac{1}{1-q^{2n}}\right)^{M_A}$$

so that $P_n(M_A) = \gamma_{W_A}(n)$. Then, by the binomial theorem, we have

$$\sum_{n=0}^{\infty} P_n(M_A)q^n = \frac{1}{2^{M_A}}\sum_{k=1}^{M_A}\binom{M_A}{k}\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{2k}\left(1-q^{2n}\right)^{M_A-k}}.$$

It suffices to find recurrence relations for each summand. Define

$$F_{M_A,k}(q) := \sum_{n=0}^{\infty} a_k(n)q^n := \prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{2k}\left(1-q^{2n}\right)^{M_A-k}}.$$

We take the logarithmic derivative of both sides as in the proof of Theorem 1. First, we take logarithms of both sides to obtain

$$\begin{aligned}\log\left(1+\sum_{n=1}^{\infty}a_k(n)q^n\right) &= -2k\sum_{n=1}^{\infty}\log\left(1-q^n\right)+(k-M_A)\sum_{n=1}^{\infty}\log\left(1-q^{2n}\right)\\ &= -(k+M_A)\sum_{n=1}^{\infty}\log\left(1-q^n\right)+(k-M_A)\sum_{n=1}^{\infty}\log\left(1+q^n\right)\\ &= (k+M_A)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{q^{mn}}{m}+(M_A-k)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^m q^{mn}}{m},\end{aligned}$$

by the Taylor expansions for $\log(1-x)$ and $\log(1+x)$. Then we take the derivatives of both sides to obtain

$$\frac{\sum_{n=1}^{\infty}na_k(n)q^{n-1}}{1+\sum_{n=1}^{\infty}a_k(n)q^n} = -\sum_{n=1}^{\infty}\sum_{d|n}d\cdot\left[(-1)^{\frac{n}{d}}(k-M_A)-(k+M_A)\right]q^{n-1},$$

so we have

$$\sum_{n=1}^{\infty}na_k(n)q^n = \left(-\sum_{n=1}^{\infty}\sum_{d|n}d\cdot\left[(-1)^{\frac{n}{d}}(k-M_A)-(k+M_A)\right]q^n\right)\left(1+\sum_{n=1}^{\infty}a_k(n)q^n\right).$$

For convenience, define $b_k(n) := \sum_{d|n}d\cdot\left[(-1)^{\frac{n}{d}}(k-M_A)-(k+M_A)\right]$. Expanding the right hand side and equating coefficients, we now have

$$0 = b_k(n)+b_k(n-1)a_k(1)+b_k(n-2)a_k(2)+\cdots+b_k(1)a_k(n-1)+na_k(n).$$

Using the same identity between the symmetric power functions and the elementary symmetric functions as in the proof of Theorem 1, we arrive at the recursion

$$\begin{aligned}a_k(n) &= \widehat{F}_n\big(a_k(1),\ldots,a_k(n-1)\big)-\frac{1}{n}\sum_{d|n}d\cdot\left[(-1)^{\frac{n}{d}}(k-M_A)-(k+M_A)\right]\\ &= \widehat{F}_n\big(a_k(1),\ldots,a_k(n-1)\big)-\sum_{\delta|n}\delta\cdot\left[(-1)^{\delta}(k-M_A)-(k+M_A)\right].\end{aligned}$$

Thus, we have

$$\begin{aligned}\gamma_{W_A}(n) = \frac{1}{2^{M_A}}\sum_{k=0}^{M_A}\binom{M_A}{k}\Bigg(&\widehat{F}_n\big(a_k(1),\ldots,a_k(n-1)\big)\\ &-\sum_{\delta|n}\delta\cdot\left[(-1)^{\delta}(k-M_A)-(k+M_A)\right]\Bigg).\end{aligned}$$

□

Remark This recurrence relation gives the coefficients $\gamma_{W_A}(n)$ in terms of the coefficients $a_k(1),\ldots,a_k(n-1)$ of each summand. Since the linear combination of infinite products is raised to the (M_A)th power in the conjugacy growth series, presumably there is no simple way to obtain a recurrence relation for $\gamma_{W_A}(n)$ in terms of $\gamma_{W_A}(1),\ldots,\gamma_{W_A}(n-1)$ as in the symmetric case.

Acknowledgements
The author would like to thank Ken Ono for his invaluable advice and guidance on this project, and also Pierre de la Harpe and the referee for their helpful comments.

Received: 16 November 2016 Accepted: 15 December 2016
Published online: 14 March 2017

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